Comparison of Elementary Fuzzy Sequential Digital Units Based on Various Popular T-norms and Co-norms

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Abstract: The fuzzy flip-flop circuit is the extended version of an ordinary, binary J-K flip-flop. The truth table of the J-K flip-flop is fuzzified, so that binary NOT, AND, and OR operations are extended to fuzzy negation, t-norm, and t-conorm, respectively. Two versions of the fundamental characteristic equation of fuzzy flip-flop were introduced in [1]. They were called 'reset type' and 'set type' equations, while both of them were fuzzy extensions of the characteristic equation of J-K flip-flop, equivalent in two valued logic but different in a fuzzy context. In this paper both versions will be used and compared and after reviewing the operations that have been investigated previously various other interesting operations will be discussed. Results of the characteristics of these new flip-flops will be demonstrated graphically in order to achieve an intuitive overview.

Keywords: t-norm, co-norm, negation, implication, fuzzy flip-flop

1 Introduction

When Zadeh first introduced fuzzy sets [2] he proposed the following operators for set complement, union, and intersection:

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \qquad (1)$$

$$\mu_{A\cup B}(x) = max[\mu_A(x), \mu_B(x)] \quad (2)$$

$$\mu_{A\cap B}(x) = min[\mu_A(x), \mu_B(x)] \quad (3)$$

In the very same original paper he mentioned in a footnote, however, that alternative definitions for the later two were possible, these alternative definitions he called interactive operators:

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x) \quad (4)$$

$$\mu_{A\cap B}(x) = \mu_A(x)\mu_B(x) \quad (5)$$

These definitions were later called “algebraic”. For each of the three set operations, a large variety of formulas and even different classes of functions possessing appropriate axiomatic properties, have subsequently been proposed. Despite this variety of fuzzy set operators, however, the original complement, union, and intersection still bear particular significance, especially in the practical applications. For instance, if the functions within a class are interpreted as performing union or intersection operations of various strengths, then the classical max union is found to be the strongest of these, and the classical min intersection the weakest.

One of the less deeply explored applications of fuzzy sets and logic is the extension of traditional Boolean logic based digital circuitry towards ‘fuzzy digital circuits’. Fuzzy gates represent no scientific challenge as they are nothing else but physical realizations of the operations themselves, except for more complex gates like XOR. It is much more interesting to examine the possibility of extension of the elementary sequential circuits, or flip-flops of which the most general one is the J-K flip-flop. In the next sections we intend to overview the results already available in the literature, including a brief summary of the concept of J-K flip-flop and then we give insight into the behavior of the various fuzzy flip-flops defined by equivalent Boolean formulations of the binary circuit and by applying a multitude of alternative definitions for the operations.

2 Binary (Boolean) and Fuzzy J-K Flip-Flops

Flip-flop circuits, especially J-K flip-flops store a single bit of information, are the basic components of every synchronous sequential digital circuit. The next state Q(t+1) of a J-K flip-flop is characterized as a function of both the present state Q(t) and two present inputs J(t) and K(t). (The truth table of a J-K flip-flop can be seen in Table 1). In the next, as a simplified notation, J, K and Q are used instead of J(t), K(t) and Q(t), respectively. The minterm expression of Q(t+1) is written as

$$Q(t+1) = \overline{J}\,\overline{K}Q + J\overline{K}\,\overline{Q} + J\overline{K}Q + JK\overline{Q} \tag{6}$$

and can be simplified as

$$Q(t+1) = J\overline{Q} + \overline{K}Q \tag{7}$$

which is well-known as the characteristic equation of J-K flip-flops. On the other hand, the equivalent maxterm expression can be given by

$$Q(t+1) = (J + Q)(\overline{K} + \overline{Q}) \tag{8}$$

J	*K*	*Q(t)*	*Q(t+1)*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Table 1
Truth table of binary J-K flip-flop

By extending the above equations of binary J-K flip-flop by substituting binary operations with their fuzzy counterparts, i.e. fuzzy negation, t-norm and t-conorm, we may change (7) and obtain (9)

$$Q_R(t+1) = (J \wedge \neg Q) \vee (\neg K \wedge Q), \tag{9}$$

further, in the same way, (8) and obtain (10)

$$Q_S(t+1) = (J \vee Q) \wedge (\neg K \vee \neg Q). \tag{10}$$

It is obvious that in fuzzy logic, equation (9) does not always equal equation (10) because in fuzzy logic the axiomatic properties are considerably weaker than in Boolean logic.

In the next section we will give an overview of the behavior of reset and set type fuzzy flip-flops (F^3) where t-norm and co-norm are various important dual fuzzy operations and the negation is the standard one defined in (1).

3 Behavior of F^3s Based on Various Fuzzy Operations

There are many different norms known from the literature which play important roles in applications or by their mathematical properties. Only very few have been investigated from the point of view of what kind of F^3 is generated by deploying these various norms in equations (9) and (10), namely the standard, algebraic norms and an interesting pair of operations generated from the standard and the £ukasiewicz operations, later proposed by Fodor. (For these three types see [1, 3], [4, 5] and [6], respectively.) From a practical aspect it is confusing that reset and set type F^3s sometimes do have very different behavior even though they are both supposed to be the extension of the same original binary circuit. Indeed they are identical at the border lines, when J and K are 0 or 1, thus these pairs are always justifiable generalizations, and nevertheless they are disturbingly non-dual. In order to eliminate this break of symmetry a combined reset-set type F^3 was proposed in [5] by the following definition:

$$Q(t+1)=\begin{cases} (J \wedge \neg Q) \vee (\neg K \wedge Q) & (J \leq K) \\ (J \vee Q) \wedge (\neg K \vee \neg Q) & (J \geq K) \end{cases} \qquad (11)$$

This general formula can be applied easily for arbitrary dual norm and co-norm, so in the next parts this combined equation will be not considered separately, except in the standard and algebraic cases, as illustrations.

The negation used throughout the whole paper will be the standard negation (1).

3.1 Standard Fuzzy Flip-Flops

In this case, equations (7) (reset type fuzzy flip-flop) and (8) (set type fuzzy flip-flop) can be expressed as

$$Q_R(t+1) = max\left[min(J,1-Q), min(1-K,Q)\right] \text{ and } \qquad (12)$$

$$Q_S(t+1) = min\left[max(J,Q), max(1-K,1-Q)\right], \qquad (13)$$

respectively.

In order to extend the binary J-K flip-flop symmetrically, the combined standard F^3 will be like this

$$Q(t+1)= \begin{cases} max[min(J,1-Q),min(1-K,Q)] \quad (J \le K) \\ min[max(J,Q),max(1-K,1-Q)] \quad (J \ge K) \end{cases} \tag{14}$$

Equation (14) is defined as the fundamental equation of the combined standard type fuzzy flip-flop.

Figure 1 shows the next states of the $Q_R(t+1)$ of the standard reset type fuzzy flip-flop for various values of J, each diagram presenting the curves for different values of Q(t) and K.

Similarly, the next states $Q_S(t+1)$ of the set type flip-flop are shown in Figure 2.

In the case of Q(t)=0 and Q(t)=1, the values of the next state Q(t+1) in both types are equal. If $0<Q(t)<1$, then Q(t+1)of the set type is always of greater or equal value than that of the reset type.

It should be noted that the values Q(t+1) of both types are continuously connected at the line segment J=K.

We should remark that all standard flip-flops are characterized by piecewise linear functions with several breakpoints in the projections, consequently break lines in the three dimensional graph. Calculation with standard operations, and consequently, with standard flip-flops is fast and easy. But their characteristic functions are not smooth.

3.2 Algebraic Fuzzy Flip-Flops

In the case of complementation for fuzzy negation, algebraic product and algebraic sum for t-norm and t-conorm, respectively, the characteristic equations (7) and (8) can be rewritten as

$$Q_R(t+1) = J + Q - 2JQ - KQ + JQ^2 + JQK - JQK^2 \tag{15}$$

$$Q_S(t+1) = J + Q - JQ - JKQ - KQ^2 + JKQ^2 \tag{16}$$

These equations show the result of transformation into a simplified form by using the definition of algebraic product and sum.

Combining equations (15) and (16) we obtain the unified equation of reset and set type:

$$Q(t+1) = J + Q - JQ - KQ \tag{17}$$

This equation is considered the fundamental equation of the algebraic fuzzy flip-flop. It is remarkable how simple this combined equation is. In addition to its simplicity it represents as symmetrical, dual solution.

The characteristics belonging to these expressions are demonstrated graphically in Figures 3 and 4.

Evaluating the curves in the Figure, they clearly demonstrate the relation between $Q_R(t+1)$ and $Q_S(t+1)$:

$$Q_S(t+1) - Q_R(t+1) = \left[J(1-K) + K(1-J)\right]Q(1-Q) \geq 0 \qquad (18)$$

thus

$$Q_R(t+1) \leq Q_S(t+1) \qquad (19)$$

as the reset types curves always go below the set ones. Algebraic operations produce smooth (differentiable) curves and surfaces with no breakpoints or lines at all. This is true also for the combined flip-flop which is not shown here.

3.3 Drastic Fuzzy Flip-Flop

It is known [7] that fuzzy set unions that satisfy the axiomatic skeleton containing boundary conditions, commutativity, monotony, and associativity are bounded by the inequalities

$$max(a,b) \leq u(a,b) \leq u_{max}(a,b), \qquad (20)$$

where

$$u_{max}(a,b) = \begin{cases} a & \text{when } b=0, \\ b & \text{when } a=0, \\ 1 & \text{otherwise.} \end{cases} \qquad (21)$$

Similarly, fuzzy set intersections satisfy

$$i_{min}(a,b) \leq i(a,b) \leq min(a,b), \qquad (22)$$

where

$$i_{min}(a,b) = \begin{cases} a & \text{when } b=1, \\ b & \text{when } a=1, \\ 0 & \text{otherwise.} \end{cases} \qquad (23)$$

Operations $u_{max}(a,b)$ and $i_{min}(a,b)$ are referred to in the literature as drastic sum and drastic product. The drastic sum and drastic product represent another pair of fuzzy union and fuzzy intersection which are extreme in the sense that for all $a,b \in [0,1]$, the inequality

$$u_{max}(a,b) - i_{min}(a,b) \geq u(a,b) - i(a,b) \tag{24}$$

is satisfied for an arbitrary pair of fuzzy union (co-norm) *u* and fuzzy intersection (t-norm) *i*.

Figures 5 and 6 show the behavior of these two extreme F^3 s. The characteristic lines are piecewise linear again the curves often following the 0 or 1 line. Obviously these lines (and the surfaces they represent realize extreme cases even for the possible flip-flops (the lines in the Figure were obtained by mechanically connecting the 5 points we had obtained for the 5 characteristic values drawn up in the picture, finer representation would lead to even more extreme curve shapes).

3.4 £ukasiewicz Fuzzy Flip-Flop

The £ukasiewicz norm and co-norm are defined as follows:

$$i_L(a,b) = max[a + b - 1,0] \tag{25}$$

$$u_L(a,b) = min[a + b,1] \tag{26}$$

Based on them the £ukasiewicz flip-flops are defined as follows:

$$Q_R(t+1) = min[max(J - Q,0) + max(Q - K,0),1] \tag{27}$$

$$Q_S(t+1) = max[min(J + Q,1) + min(2 - K - Q,1) - 1,0] \tag{28}$$

These sets of characteristics are identical with the characteristics of the Yager reset type flip-flop (w =1) in Figures 8 and 9.

3.5 Fodor Fuzzy Flip-Flop (F^4)

In [6] Fodor and Kóczy proposed non-associative operations for a new class of fuzzy flip-flops. It was stated there that any F^3 satisfying:

P1: $F_i(0, 0, Q) = Q$,

P2: $F_i(0, 1, Q) = 0$,

P3: $F_i(1, 0, Q) = 1$,

P4: $F_i(1, 1, Q) = n_i(Q)$.

P5: $F_i(\nu_i, \nu_i, Q) = \nu_i$.

P6: $F_i(D, n_i(D), Q) = D$,

Where $\nu_i = n_i(\nu_i)$ for i=1, 2.

Is a ϕ-transform of the basic algebraic F^3 as given in Section 3.2, were the ϕ-transform is an automorphism of the unit interval such that

$$Q_R(t+1) = \phi^{-1}\left[\phi(J)(1-\phi(Q))+\phi(Q)(1-\phi(K))\right]. \quad (29)$$

Similarly, for the ψ -transform

$$Qs(t+1) = \psi^{-1}\left[\psi(J)(1-\psi(Q))+\psi(Q)(1-\psi(K)\right]. \quad (30)$$

Thus another solution of P6:

$$Q_R(t+1) = \frac{min\{J,1-Q\}+max\{J-Q,0\}}{2}+$$

$$\frac{min\{Q,1-K)\}+max\{Q-K,0\}}{2}, \quad (31)$$

$$Q_S(t+1) = \frac{max\{J,Q\}+max\{J+Q-1,0\}}{2}+$$

$$\frac{max\{1-Q,1-K\}+max\{1-Q-K,0\}}{2} \quad (32)$$

These equations were obtained by combining the standard and the £ukasiewicz norms by the arithmetic mean in the inner part of the formula. The other parts use £ukasiewicz operations. When recalculating (31) and (32) we find however that the second formula is incorrect in [6] because:

$$Q_R(t+1) = min\left[T(J,1-Q)+T(1-K,Q),1\right] =$$

$$= T(J,1-Q)+T(1-K,Q) \quad \text{and} \quad (33)$$

$$Q_S(t+1) = max\left[S(J,Q)+S((1-K),(1-Q))-1,0\right] =$$

$$= S(J,Q)+S(1-K,1-Q)-1 \quad (34)$$

where T and S denote the £ukasiewicz norms. From here:

$$Q_S(t+1)=\frac{max(J,Q)+max(1-K,1-Q)-1}{2}+$$

$$\frac{min(J+Q,1)+min(1-K+1-Q,1)-1}{2}=$$

$$=\frac{max(J,Q)+max(1-K,1-Q)}{2}+$$

$$\frac{min(J+Q,1)+min(2-K-Q,1)}{2}-1 \qquad (35)$$

Comparing this corrected form of the set type F^4 we came to the surprising result, that there is only one F^4 in this particular case as the two formulas are equivalent. Figure 7 presents the characteristic diagrams for these singular non-associative F^3. Indeed, the diagram shows a similarity with the algebraic case, however with luck of smoothness.

3.6 Yager and Dombi Fuzzy Flip-Flops

Several classes of functions have been proposed whose individual members satisfy all the axiomatic requirements for the fuzzy union and neither, one, or both of the optional axioms. One of these classes of fuzzy unions is known as the *Yager class* and is defined by the function:

$$u_w(a,b)=min\left[1,(a^w+b^w)^{1/w}\right], \qquad (36)$$

where values of the parameter *w* lie within the open interval $(0,\infty)$.

Yager intersection formula is given by:

$$i_w(a,b)=1-min\left[1,((1-a)^w+(1-b)^w)^{1/w}\right]. \qquad (37)$$

In the formulas, *a* represents the membership grade for an element in fuzzy set A, and *b* represents the membership grade for an element in fuzzy set B.

Using these definitions we can find the solution for:

$$Q_R(t+1)=u_w\left[i_w(J,1-Q),i_w(1-K,Q)\right] \text{ and} \qquad (38)$$

$$Q_S(t+1)=i_w\left[u_w(J,Q),u_w(1-K,1-Q)\right]. \qquad (39)$$

Figures 8, 9, 10, 11 present the diagrams for Yager flip-flops, for two typical cases when w = 1 and w = 4. The real curves are here also smooth.

Finally, the following equations, where values of the parameter α lie within the open interval $(0,\infty)$ are called Dombi operators:

$$u_\alpha(a,b)=\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{-\alpha}+\left(\frac{1}{b}-1\right)^{-\alpha}\right]^{-1/\alpha}} \quad (40)$$

$$i_\alpha(a,b)=\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{\alpha}+\left(\frac{1}{b}-1\right)^{\alpha}\right]^{1/\alpha}} \quad (41)$$

The Dombi F^3 is defined as:

$$Q_R(t+1)=u_\alpha\left[i_\alpha(J,1-Q),i_\alpha(1-K,Q)\right] \text{ and} \quad (42)$$

$$Q_S(t+1)=i_\alpha\left[u_\alpha(J,Q),u_\alpha(1-K,1-Q)\right]. \quad (43)$$

Figures 12, 13, 14 and 15 depict typical values in the Dombi operator case for two typical cases when α = 1, and α = 4.

If $a = 0$ or $b = 0$ the value is understood in limit. These curves are also smooth.

Conclusions

We have examined a multitude of fuzzy operations from the point of view of how they generate reset and set types F^3s. These different flip-flops showed interesting behaviour, sometimes having breakpoints and sometimes being smooth. These are some general topological behaviours which form certain classes producing monotonic, convex or concave and sometimes sigmoidal functions, as segments of the flip-flops' functional surfaces. We found a single case when the two flip-flops merge into a single one; these were the F^4 based on the corrected formula given in [6]. It remains an interesting open question, whether there are any other non-associative operations which have the same advantageous properties as F^4, which needs further investigations.

Acknowledgement

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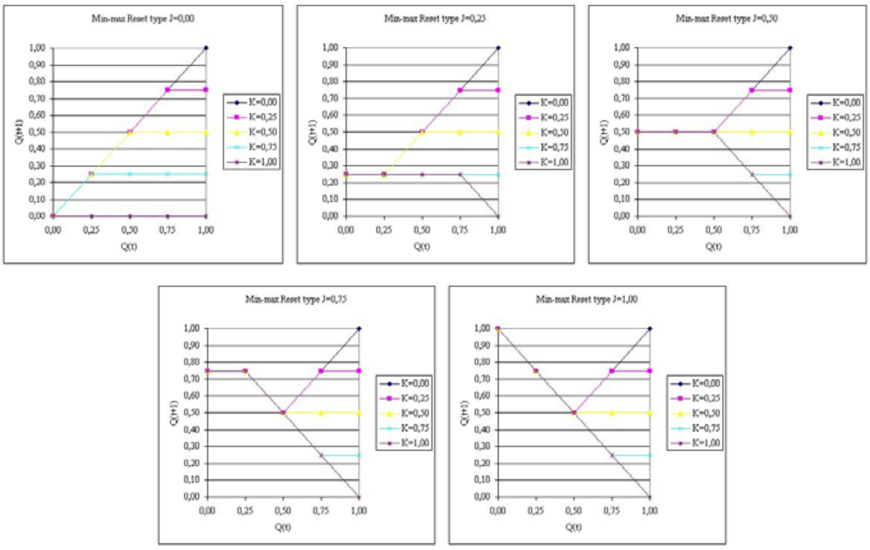

Figure 1

Characteristics of min-max reset type fuzzy flip-flop for various values of J

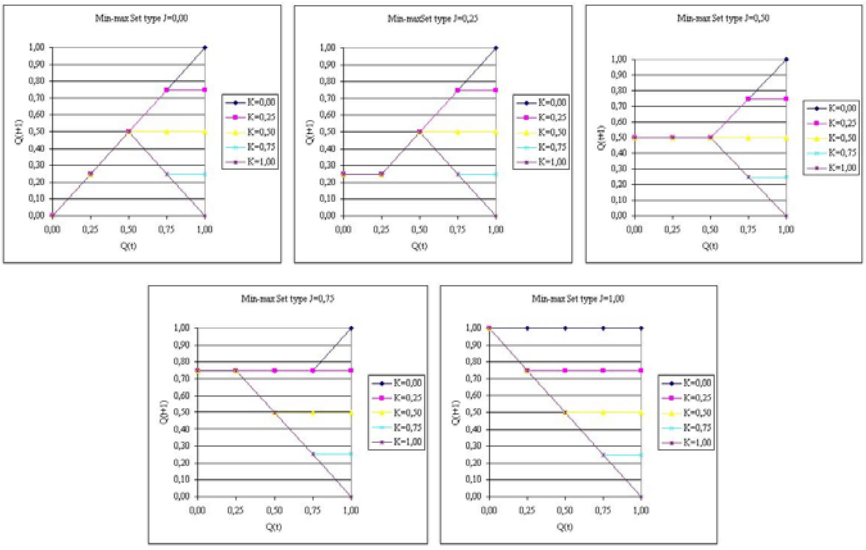

Figure 2

Characteristics of min-max set type fuzzy flip-flop for various values of J

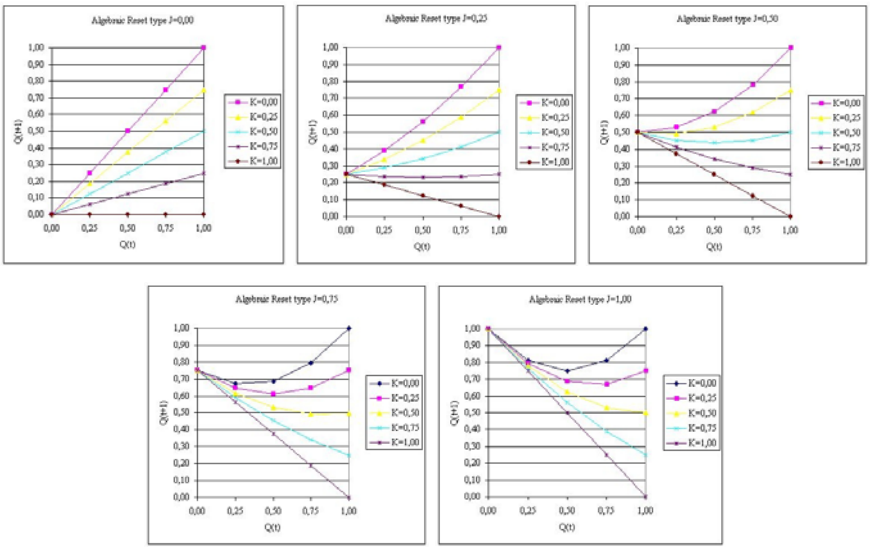

Figure 3
Characteristics of algebraic reset type fuzzy flip-flop for various values of J

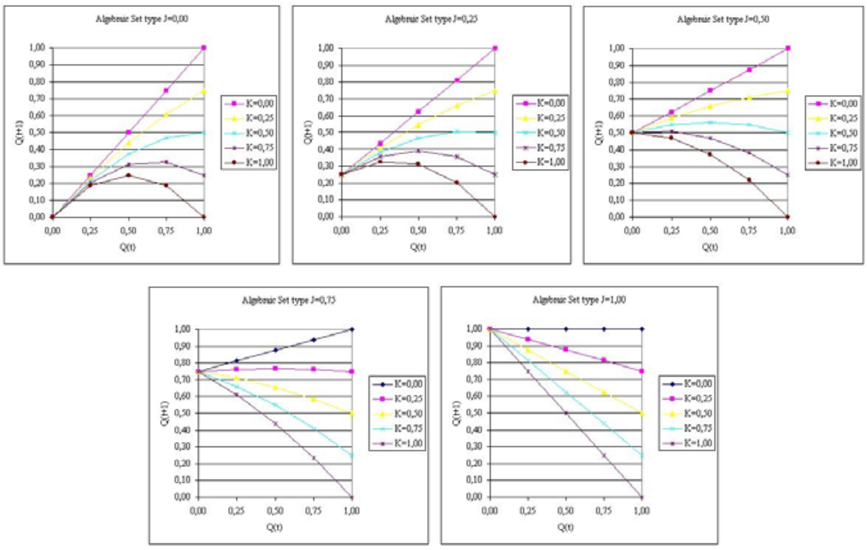

Figure 4
Characteristics of algebraic set type fuzzy flip-flop for various values of J

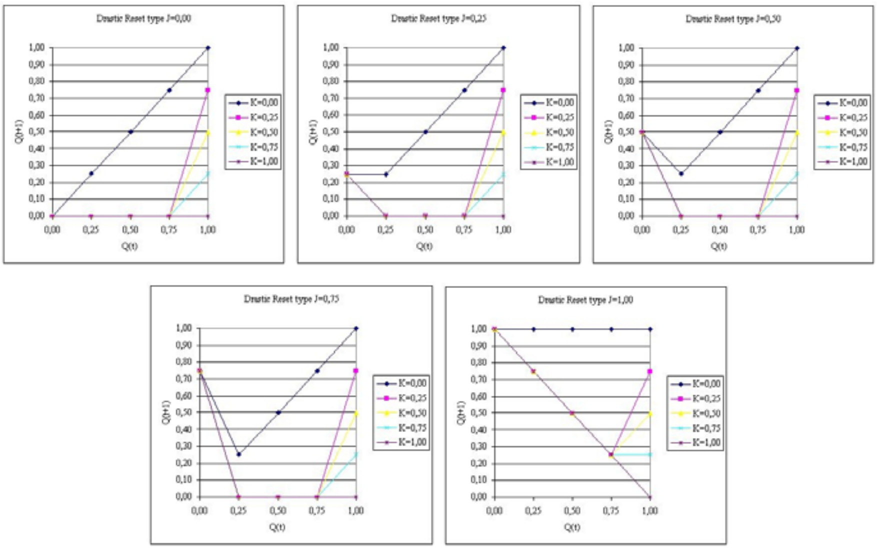

Figure 5
Characteristics of drastic reset type fuzzy flip-flop for various values of J

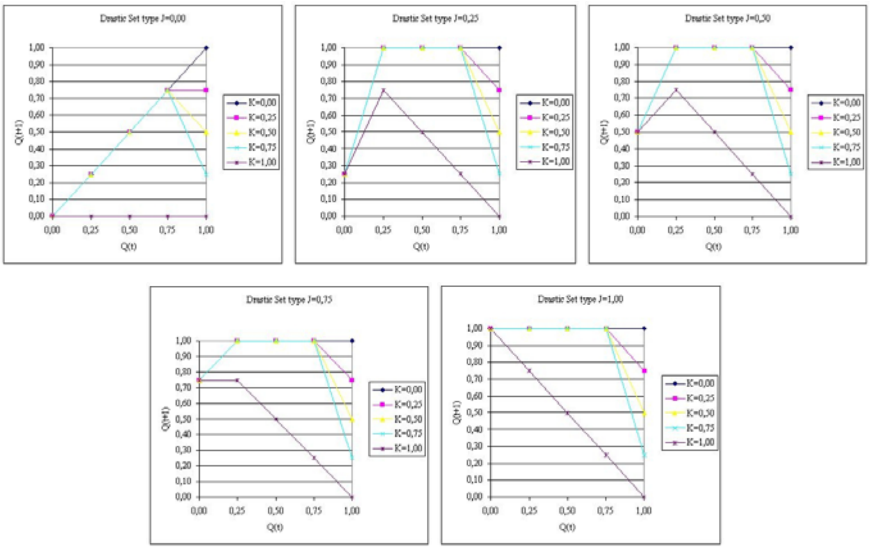

Figure 6
Characteristics of drastic set type fuzzy flip-flop for various values of J

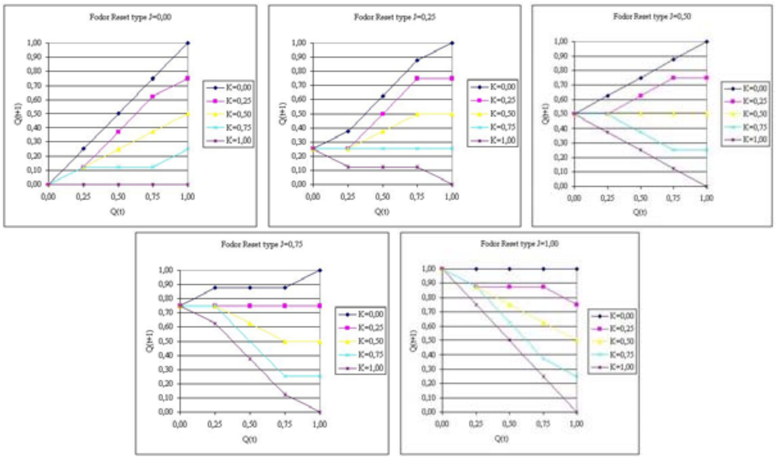

Figure 7

Characteristics of Fodor reset type fuzzy flip-flop for various values of J

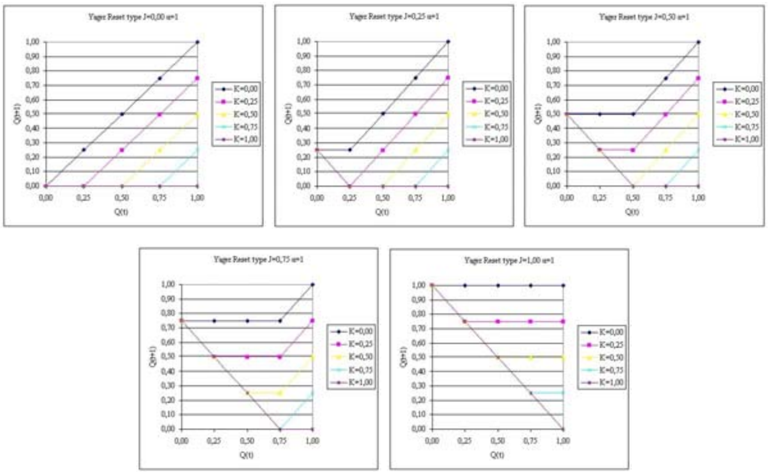

Figure 8

Characteristics of Yager reset type fuzzy flip-flop for various values of J (w=1)

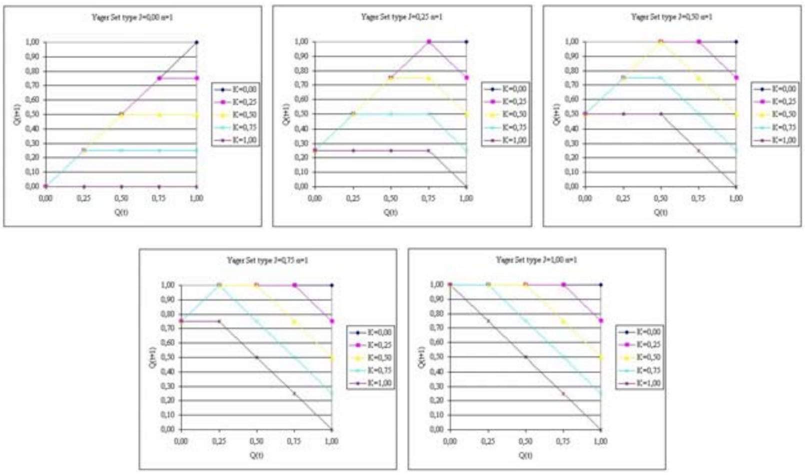

Figure 9
Characteristics of Yager set type fuzzy flip-flop for various values of J (w=1)

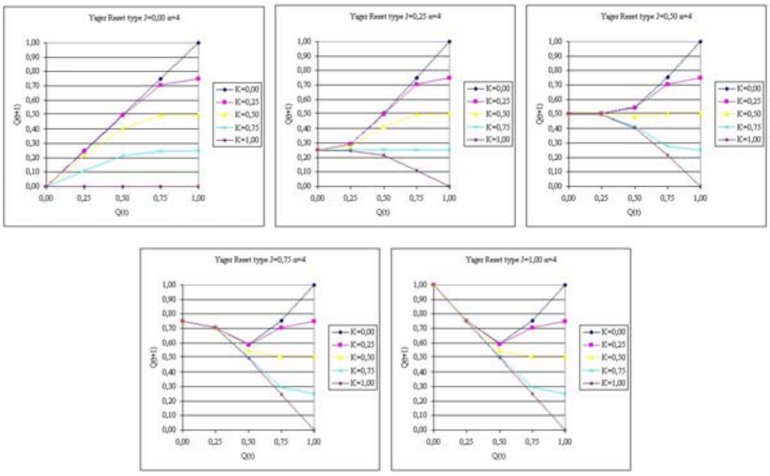

Figure 10
Characteristics of Yager reset type fuzzy flip-flop for various values of J (w=4)

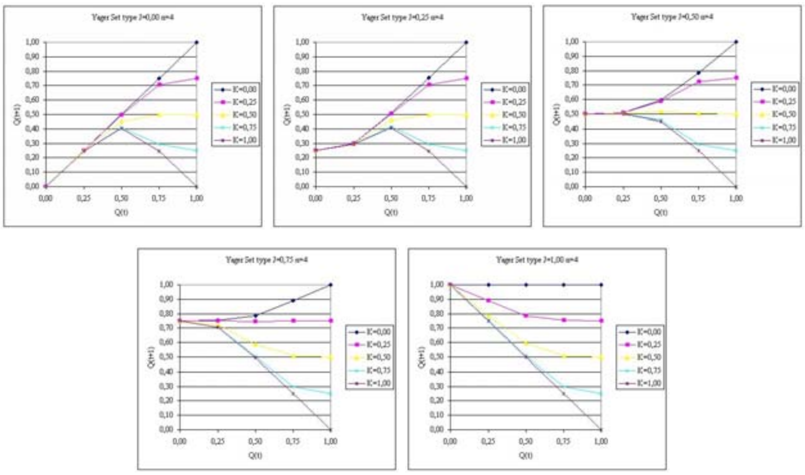

Figure 11

Characteristics of Yager set type fuzzy flip-flop for various values of J (w=4)

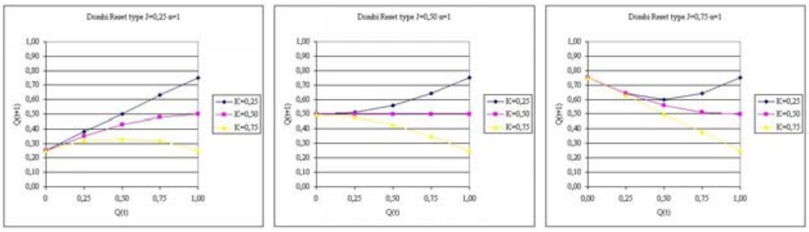

Figure 12

Characteristics of Dombi reset type fuzzy flip-flop for various values of J (α=1)

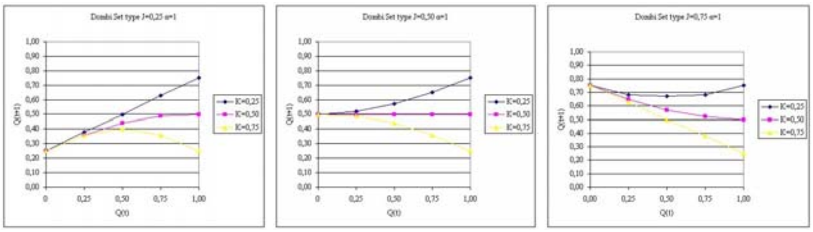

Figure 13

Characteristics of Dombi set type fuzzy flip-flop for various values of J (α=1)

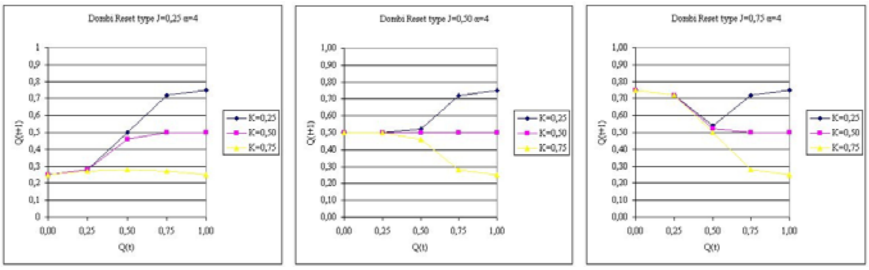

Figure 14
Characteristics of Dombi reset type fuzzy flip-flop for various values of J (α=4)

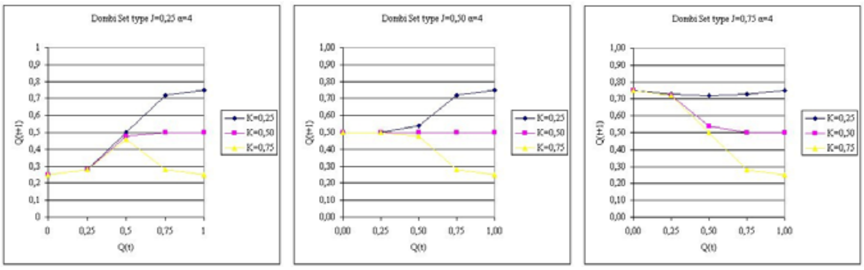

Figure 15
Characteristics of Dombi set type fuzzy flip-flop for various values of J (α=4)