Arithmetics with Fuzzy Numbers: a Comparative Overview

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Abstract: It is well known that the result of some extension principle-based arithmetic operations on trapezoidal fuzzy numbers is not trapezoidal. To avoid this shortcoming two main approaches exist. The first one tries to approximate the resulting fuzzy number with an appropriate trapezoidal one. The second one re-defines the arithmetic operations in a way that results in trapezoidal fuzzy numbers directly. The aim of the present paper is to give an overview of such recent approaches. At the end we compare the methods and give some conclusion.

Keywords: fuzzy number, extension principle, trapezoidal approximation, cross product

1 Introduction

Fuzzy numbers are fuzzy subsets of the set of real numbers satisfying some additional conditions. Fuzzy numbers allow us to modell non-probabilistic uncertainties in an easy way. Arithmetic operations on fuzzy numbers have also been developed, and are based mainly on the extension principle [15] or on interval arithmetic [11]. When operating with fuzzy numbers, the result of our calculations strongly depend on the shape of the membership functions of these numbers. Less regular membership functions lead to more complicated calculations. Moreover, fuzzy numbers with simpler shape of membership functions often have more intuitive and more natural interpretation.

Thus, there is a natural need of simple approximations of fuzzy numbers that are easy to handle and have a natural interpretation. Trapezoidal or triangular fuzzy numbers are most common in current applications.

Considering the extension principle-based arithmetic operations on trapezoidal fuzzy numbers, the product of two such fuzzy numbers is not of the same kind: the shape of these fuzzy numbers is not preserved. In many situations this problem is solved by approximating the result of the extension principle-based multiplication by a triangular or trapezoidal number. In some other cases the arithmetical operations are defined in a way that the results are always trapezoidal fuzzy numbers.

The aim of the present paper is to give an overview of recent approaches to these problems. After the necessary preliminaries we consider two groups of solutions. Methods in the first one try to approximate a fuzzy number with a trapezoidal one. Approaches in the second group re-define the arithmetic operations in a way that results in trapezoidal fuzzy numbers directly. At the end we compare the methods and give some conclusion.

2 Preliminaries

Let us recall the following well-known definition of a fuzzy number. The addition of fuzzy numbers and multiplication of a fuzzy number by a crisp number are provided by Zadeh's extension principle.

Definition 1 A fuzzy number is a function $u : \mathbf{R} \to [0,1]$ with the following properties:

(i) u is normal, i.e., there exists $x_0 \in \mathbf{R}$ such that $u(x_0) = 1$;

(ii) $u(\lambda x + (1-\lambda) y) \geq \min\{u(x), u(y)\}, \forall x, y \in R, \forall \lambda \in [0,1]$;

(iii) u is upper semicontinuous on $\mathbf{R}$, i.e., $\forall x_0 \in \mathbf{R}$ and $\forall \varepsilon > 0$ there exists a neighborhood $V(x_0)$ such that $u(x) \leq u(x_0) + \varepsilon, \forall x \in V(x_0)$;

(iv) The set $\overline{\operatorname{supp}(u)}$ is compact in $\mathbf{R}$, where $\operatorname{supp}(u) = \{x \in R; u(x) > 0\}$.

We denote by R_F the set of all fuzzy numbers.

Let $a,b,c \in \mathbf{R}, a < b < c$. The fuzzy number $u : \mathbf{R} \to [0,1]$ denoted by (a,b,c) and defined by $u(x) = 0$ if $x \leq a$ or $x \geq c, u(x) = \frac{x-a}{b-a}$ if $x \in [a,b]$ and $u(x) = \frac{c-x}{c-b}$ if $x \in [b,c]$ is called a triangular fuzzy number.

For $0 < r \le 1$ and $u \in R_F$ we denote $[u]^r = \{x \in \mathbf{R}; u(x) \ge r\}$ and $[u]^0 = \overline{\{x \in \mathbf{R}; u(x) > 0\}}$. It is well-known that for each $r \in [0,1], [u]^r$ is a bounded closed interval, $[u]^r = [\underline{u}^r, \overline{u}^r]$. Let $u, v \in R_F$ and $\lambda \in \mathbf{R}$. We define the sum $u + v$ and the scalar multiplication λu by

$$[u+v]^r = [u]^r + [v]^r = [\underline{u}^r + \underline{v}^r, \overline{u}^r + \overline{v}^r]$$

and

$$[\lambda u]^r = \lambda [u]^r = \begin{cases} [\lambda \underline{u}^r, \lambda \overline{u}^r], & \text{if } \lambda \ge 0, \\ [\lambda \overline{u}^r, \lambda \underline{u}^r], & \text{if } \lambda < 0, \end{cases}$$

respectively, for every $r \in [0,1]$.

We denote by $-u = (-1)u \in R_F$ the symmetric of $u \in R_F$.

The product $u \cdot v$ of fuzzy numbers u and v, based on Zadeh's extension principle, is defined by

$$\underline{(u \cdot v)}^r = \min\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}$$
$$\overline{(u \cdot v)}^r = \max\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}.$$

Surely, the above formulas are not very practical from the computational point of view. Also, let us remark that usually the fuzzy numbers which are used in practical applications are trapezoidal. So, the requirement that a product operation should be shape-preserving seems to be natural.

Definition 2 A fuzzy number $u \in R_F$ is said to be positive if $\underline{u}^1 \ge 0$, strict positive if $\underline{u}^1 > 0$, negative if $\overline{u}^1 \le 0$ and strict negative if $\overline{u}^1 < 0$. We say that u and v have the same sign if they are both positive or both negative.

Let $u, v \in R_F$. We say that $u \prec v$ if $\underline{u}^r \le \underline{v}^r$ and $\overline{u}^r \le \overline{v}^r$ for all $r \in [0,1]$. We say that u and v are on the same side of 0 if $u \prec 0$ and $v \prec 0$ or $0 \prec u$ and $0 \prec v$.

Remark 1 If u is positive (negative) then $-u$ is negative (positive).

Definition 3 For arbitrary fuzzy numbers u and v the quantity

$$D(u,v) = \sup_{0 \le r \le 1} \{\max\{|\underline{u}^r - \underline{v}^r|, |\overline{u}^r - \overline{v}^r|\}\}$$

is called the (Hausdorff) distance between u and v.

It is well-known (see e.g. [3]) that (R_F, D) is a complete metric space and D verifies $D(ku,kv)=|k|D(u,v)$, $\forall u,v\in R_F$, $\forall k\in R_F$.

The so-called L - R fuzzy numbers are considered important in fuzzy arithmetic. These and their particular cases triangular and trapezoidal fuzzy numbers are used almost exclusively in applications.

Definition 4 ([4], p. 54, [3]) Let L,R : $[0,+\infty)\to[0,1]$ be two continuous, decreasing functions fulfilling $L(0)=R(0)=1, L(1)=R(1)=0$, invertible on $[0,1]$. Moreover, let a^1 be any real number and suppose $\underline{a},\overline{a}$ be positive numbers. The fuzzy set $u:\mathbf{R}\to[0,1]$ is an L - R fuzzy number if

$$u(t)=\begin{cases} L\left(\frac{a^1-t}{\underline{a}}\right), & \text{for } t\le a^1 \\ R\left(\frac{t-a^1}{\overline{a}}\right), & \text{for } t>a^1. \end{cases}$$

Symbolically, we write $u=\left(a^1,\underline{a},\overline{a}\right)_{L,R}$, where a^1 is called the mean value of u, $\underline{a},\overline{a}$ are called the left and the right spread. If u is an L - R fuzzy number then (see e. g. [13])

$$[u]^r=\left[a^1-L^{-1}(r)\underline{a},a^1+R^{-1}(r)\overline{a}\right].$$

As a particular case, one obtains *trapezoidal fuzzy numbers* when the functions L and R are linear. A trapezoidal fuzzy number u can be represented by the quadruple $(a,b,c,d)\in\mathbf{R}^4$, $a\le b\le c\le d$. In this case the r -level sets are given by $\underline{u}^r=a+r(b-a)$ and $\overline{u}^r=d+r(d-c)$. If we have $b=c$ in the representation (a,b,c,d), the fuzzy number is called *triangular*. Then we can use the triple (a,b,d) only.

A trapezoidal fuzzy number (a,b,c,d) can also be represented by a quadruple $\langle m,U,L,R\rangle$, where $m=(b+c)/2$ is the *modal value*, $U=m-c$ is the *upper tolerance*, and $L=m-a$ and $R=d-m$ are the *left and right lower tolerances* respectively (see Figure 1).

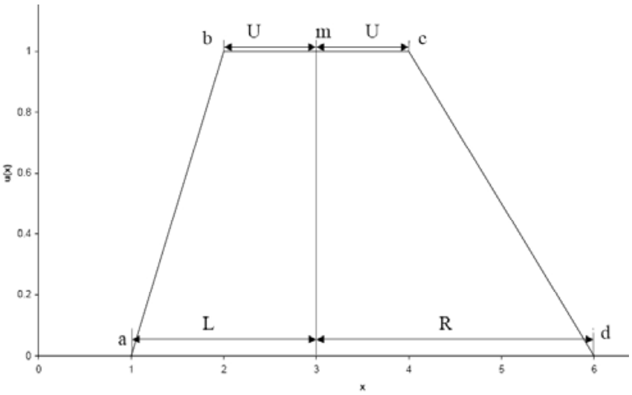

Figure 1
Two representations of a trapezoidal fuzzy number

The r -level sets of a trapezoidal fuzzy number $u = \langle m_u, U_u, L_u, R_u \rangle$ are given as follows:

$$[u]^r = [m_u - L_u + r(L_u - U_u), m_u + R_u + r(U_u - R_u)].$$

3 Trapezoidal Approximations

3.1 Conventional Fuzzy Arithmetic with Trapezoidal Fuzzy Numbers

In conventional fuzzy arithmetic with $u, v \in R_F$, the arithmetic operations $\circ \in \{+, -, \cdot, \div\}$ are defined by applying interval arithmetic to the r -level sets $[u]^r = \left[\underline{u}^r, \overline{u}^r\right]$ and $[v]^r = \left[\underline{v}^r, \overline{v}^r\right]$ of the fuzzy numbers. The sum and the difference of two trapezoidal fuzzy numbers are also trapezoidal. The product and the quotient are, however, of non-trapezoidal shape.

As an example, consider $u = (0, 2, 4, 6)$ and $v = (2, 3, 8)$. Then $u + v = (2, 5, 7, 14)$ and $[u \cdot v]^1 = [6, 12]$, $[u \cdot v]^0 = [0, 48]$, with membership function shown in Figure 2.

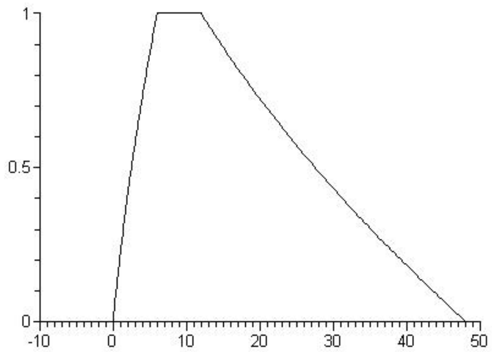

Figure 2
Actual product of (0,2,4,6) and (2,3,8)

3.2 Trapezoidal Approximation of the Conventional Product

In applications which use trapezoidal fuzzy numbers we often opt for computational simplicity and calculate only the core $[u \cdot v]^1 = (6,12)$ and the support $[u \cdot v]^0 = (0,48)$ using interval methods. The resultant trapezoidal fuzzy number $(0,6,12,48)$ is then used for the approximation of the actual product in Figure 2.

Repeated operations on fuzzy numbers using conventional fuzzy arithmetic may have the effect of increasing the uncertainty with each successive operation. This is inconsistent with the day to day experience and the intuitive way people handle vague quantities [9].

3.3 Trapezoidal Approximation and Least Squares Fitting

Turning back to the just mentioned trapezoidal approximation of the product of two trapezoidal fuzzy numbers, a very natural idea is to use linear approximations of side functions based on classical least squares. Without going into details, it is interesting to notice that the slopes of these least squares approximations are equal to the respective slopes of the trapezoidal approximation determined by the core and the support of the product. For more details and other results we refer to [5].

3.4 New Trapezoidal Approximation Preserving the Expected Interval

As we have discussed in the previous sections, the usual (Zadeh's extension principle-based) product does not preserve the shape of the operands. Thus, the result of the product, for computational purposes has to be approximated by trapezoidal number. A new, axiomatic approach has been introduced in [8]. We will call the trapezoidal approximation of the product based on this method as new trapezoidal approximation of the product.

The method proposed in [8] gives the best approximation of the product under some appropiate conditions. These conditions are natural, so in the approximation of the product the result obtained is motivated from the theoretical point of view. Let us regard the trapezoidal approximation as an operator T, $T : R_F \to R_F$, which for a given fuzzy number u gives its trapezoidal approximation. The list of requirements which have to be satisfied by an operator of this type are given in [8] below.

1 Conserving some fixed α -cut. E.g. if the operator preserves the 0 and 1 level sets, then the old trapezoidal approximation is reobtained.

2 Invariance to translation of the operator T.

3 Invariance with respect to rescaling.

4 Monotonicity with respect to inclusion.

5 Idempotency (i.e. the trapezoidal approximation of a trapezoidal number is itself).

6 To be best approximation, that is, it should be the nearest in some prescribed sence ($D(T(u),u) \leq D(x,u)$ for any trapezoidal number x).

7 Conserves the so-called expected interval, that is the original fuzzy number and its approximation has the same expected interval (Let us recall here that the expected interval of $u \in R_F$ is $\left[\int_0^1 \underline{u}^r dr, \int_0^1 \overline{u}^r dr\right]$).

8 Continuity.

9 Compatiblity with the extension principle.

10 Monotonicity with respect to some ordering between fuzzy numbers.

11 Invariance with respect to correlation. (see [7]).

In [8] the authors propose a trapezoidal approximation which is best approximation and preserves the expected interval, that is conditions 6 and 7 are required. In this case, for $u \in R_F$ we obtain the trapezodal fuzzy number (t_1, t_2, t_3, t_4), where

$$t_1 = -6\int_0^1 r\underline{u}^r dr + 4\int_0^1 \underline{u}^r dr,$$

$$t_2 = 6\int_0^1 r\underline{u}^r dr - 2\int_0^1 \underline{u}^r dr,$$

$$t_3 = 6\int_0^1 r\overline{u}^r dr - 2\int_0^1 \overline{u}^r dr,$$

$$t_4 = -6\int_0^1 r\overline{u}^r dr + 4\int_0^1 \overline{u}^r dr.$$

In [8], the authors proved also that conditions 2,3,4,5,8,9,10,11 are fulfiled. Moreover, the expected value of the fuzzy numbers (called also defuzzification by the center of area method) is preserved.

4 New Operations Resulting in Trapezoidal Fuzzy Numbers

4.1 The Cross Product

In this section we study the theoretical properties of the cross product of fuzzy numbers. For more details see [1] and [2].

Let $R_F^* = \{u \in R_F \ : \ u \text{ is positive or negative}\}$. Firstly we begin with a theorem which was obtained by using the stacking theorem ([12]).

Theorem 1 *If u and v are positive fuzzy numbers then $w = u \odot v$ defined by $[w]^r = \left[\underline{w}^r, \overline{w}^r\right]$, where $\underline{w}^r = \underline{u}^r \underline{v}^1 + \underline{u}^1 \underline{v}^r - \underline{u}^1 \underline{v}^1$ and $\overline{w}^r = \overline{u}^r \overline{v}^1 + \overline{u}^1 \overline{v}^r - \overline{u}^1 \overline{v}^1$, for every $r \in [0,1]$, is a positive fuzzy number.*

Corollary 1 *Let u and v be two fuzzy numbers.*

(i) *If u is positive and v is negative then $u \odot v = -(u \odot (-v))$ is a negative fuzzy number;*

(ii) *If u is negative and v is positive then $u \odot v = -((-u) \odot v)$ is a negative fuzzy number;*

(iii) *If u and v are negative then $u \odot v = (-u) \odot (-v)$ is a positive fuzzy number.*

Definition 5 The binary operation $\odot$ on R_F^* introduced by Theorem 1 and Corollary 1 is called cross product of fuzzy numbers.

Remark 1) The cross product is defined for any fuzzy numbers in

$R_F^{\wedge} = \left\{ u \in R_F^{*};\right.$ there exists an unique $x_0 \in \mathbf{R}$ such that $\left. u\left(x_0\right) = 1\right\}$, so implicitly for any triangular fuzzy number. In fact, the cross product is defined for any fuzzy number in the sense proposed in [6] (see also [13]).

2) The below formulas of calculus can be easily proved ($r \in [0,1]$):

$$\underline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1,$$
$$\overline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1$$

if u is positive and v is negative,

$$\underline{(u \odot v)}^r = \underline{u}^r \overline{v}^1 + \underline{u}^1 \overline{v}^r - \underline{u}^1 \overline{v}^1,$$
$$\overline{(u \odot v)}^r = \overline{u}^r \underline{v}^1 + \overline{u}^1 \underline{v}^r - \overline{u}^1 \underline{v}^1$$

if u is negative and v is positive. In the last possibility, if u and v are negative then

$$\underline{(u \odot v)}^r = \overline{u}^r \overline{v}^1 + \overline{u}^1 \overline{v}^r - \overline{u}^1 \overline{v}^1,$$
$$\overline{(u \odot v)}^r = \underline{u}^r \underline{v}^1 + \underline{u}^1 \underline{v}^r - \underline{u}^1 \underline{v}^1.$$

3) The cross product extends the scalar multiplication of fuzzy numbers. Indeed, if one of operands is the real number k identified with its characteristic function then $\underline{k}^r = \overline{k}^r = k, \forall r \in [0,1]$ and following the above formulas of calculus we get the result.

The following interpretation related to error theory is a further theoretical motivation of the use of the cross product of fuzzy numbers. Indeed, the consistency of the cross product with the classical theory motivates its use in the case of modelling uncertain data (uncertainty being due to errors of measurement).

We introduce two kinds of errors of fuzzy numbers corresponding to absolute error and relative error in classical error theory and we study these with respect to sum and cross product.

Definition 6 Let u be a fuzzy number. The crisp number $\Delta_L^r(u) = \underline{u}^1 - \underline{u}^r$ is called r - error to left of u and the crisp number $\Delta_R^r(u) = \overline{u}^r - \overline{u}^1$ is called r - error to right of u, where $r \in [0,1]$. The sum $\Delta^r(u) = \Delta_L^r(u) + \Delta_R^r(u)$ is called r -error of u.

If u expresses the fuzzy concept A then $\Delta_L^r(u)$ and $\Delta_R^r(u)$ can be interpreted as the values of tolerance of level r from the concept A to left and to right, respectively. For example, if the triangular fuzzy number $u=(5,7,9)$ expresses "early morning" then $\Delta_L^{\frac{1}{2}}(u)=1$ (one hour) is the tolerance of level $\frac{1}{2}$ of u towards night from the concept of "early morning" and $\Delta_R^{\frac{1}{4}}(u)=0.5$ (30 minutes) is the tolerance of level $\frac{1}{4}$ of u towards moon from the concept of "early morning".

A new argument in the use of addition of fuzzy numbers as extension (by Zadeh's principle) of real addition is the validity of the formula

$$\Delta^r(u+v)=\Delta^r(u)+\Delta^r(v),$$

which is consistent to the classical error theory. It is an immediate consequence of the obvious formulas

$$\Delta_L^r(u+v)=\Delta_L^r(u)+\Delta_L^r(v)$$

and

$$\Delta_R^r(u+v)=\Delta_R^r(u)+\Delta_R^r(v).$$

Now, let us study the relative error of the cross product.

Definition 7 Let u be a fuzzy number such that $\underline{u}^1 \neq 0$ and $\overline{u}^1 \neq 0$. The crisp numbers $\delta_L^r(u)=\frac{\Delta_L^r(u)}{|\underline{u}^1|}$ and $\delta_R^r(u)=\frac{\Delta_R^r(u)}{|\overline{u}^1|}$ are called relative r-errors of u to left and to right. The quantity $\delta^r(u)=\delta_L^r(u)+\delta_R^r(u)$ is called relative r-error of u.

Theorem 5 *If u and v are strict positive or strict negative fuzzy numbers then*

$$\delta^r(u\odot v)=\delta^r(u)+\delta^r(v).$$

Corollary 2 *If u is a strict positive fuzzy number then $\delta_L^r(u^{\odot n})=n\delta_L^r(u)$, $\delta_R^r(u^{\odot n})=n\delta_R^r(u)$ and $\delta^r(u^{\odot n})=n\delta^r(u)$.*

The above theorems show us that the cross product is consistent with the classical error theory (the propagation of errors is governed by a similar law as in the classical case).

4.2 New Arithmetic Operations on Trapezoidal Fuzzy Numbers

In [10] a new set of arithmetic operations was introduced for $L-R$ fuzzy numbers. Now we consider its specification for trapezoidal fuzzy numbers. We will employ representation (ma1) here.

Let $u=\langle m_u, U_u, L_u, R_u\rangle$ and $v=\langle m_v, U_v, L_v, R_v\rangle$ be two trapezoidal fuzzy numbers and consider any arithmetic operation $\circ \in \{+,-,\cdot,\div\}$. By definition from [10],

$$m_{u\circ v} = m_u \circ m_v,$$

and

$$[m_{u\circ v}]^r = [m_u \circ m_v - L_r(u,v), m_u \circ m_v + R_r(u,v))$$

with

$$L_r(u,v) = \max\{L_u - r(L_u - U_u), L_v - r(L_v - U_v)\},$$

$$R_r(u,v) = \max\{R_u - r(R_u - U_u), R_v - r(R_v - U_v)\}.$$

This definition is illustrated in Figure 3, using fuzzy numbers $u=(0,2,4,6)$ and $v=(2,3,8)$ as before.

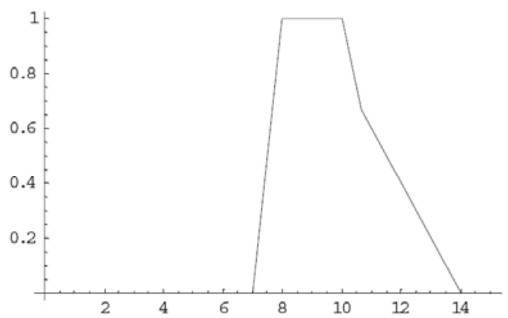

Figure 3
Multiplication of two trapezoidal fuzzy numbers using definition from [10]

As one can see easily, the result is not trapezoidal. Its side functions are piecewise linear in general.

4.3 Simplified Operations on Trapezoidal Fuzzy Numbers

In [14], a simplification of Ma et al's definition was published. Essentially, it is just the traditional trapezoidal approximation applied for Ma et al's result. More formally, let $u = \langle m_u, U_u, L_u, R_u \rangle$ and $v = \langle m_v, U_v, L_v, R_v \rangle$ be two trapezoidal fuzzy numbers and consider any arithmetic operation $\circ \in \{+, -, \cdot, \div\}$. Then define the extension of $\circ$ as follows:

$$u \circ v = \langle m_u \circ m_v, \max\{U_u, U_v\}, \max\{L_u, L_v\}, \max\{R_u, R_v\} \rangle.$$

5 Comparisons

Now we compare four approaches outlined above: i) actual product (based on the extension principle), ii) old trapezoidal approximation of the actual product, iii) the cross product, iv) new trapezoidal approximation of the actual product. Trapezoidal fuzzy numbers $(1,5,8,10)$, $(1,3,4,8)$ are considered, and the four cases are illustrated in Figure 4. The solid vertical line and the dashed vertical line represent the defuzzified values of the results by centroid and expected values (center of area) methods, respectively.

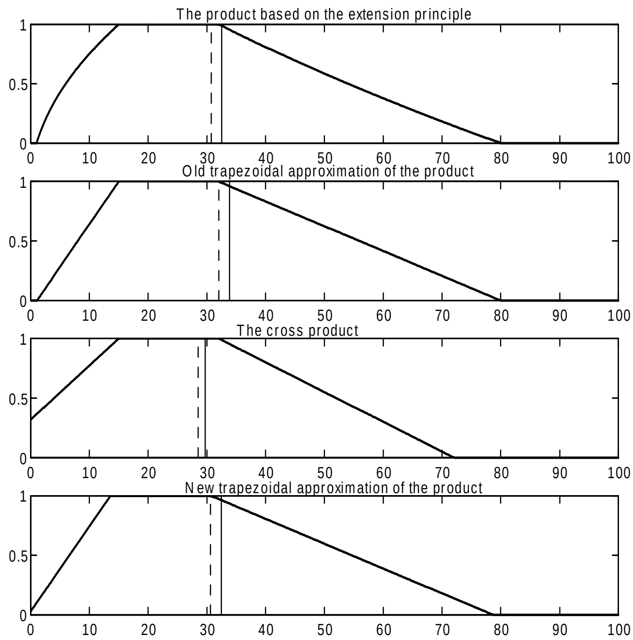

Figure 4
The product of two trapezoidal fuzzy numbers obtained by the different approaches discussed in the paper

Figure 4 does not show a striking difference between the results of the different methods. However, the difference can be significant if we perform iterative computations with fuzzy numbers. In order to show this we consider the exponential functions obtained as power series with respect to the product type operations discussed above. In Figure 5 the results of the exponential-type functions are presented. The fuzzy number considered in the exponent is $(2.2, 4.6, 4.7, 5)$. Solid thin line represent again defuzzification by centroid method, while dashed line the expected value.

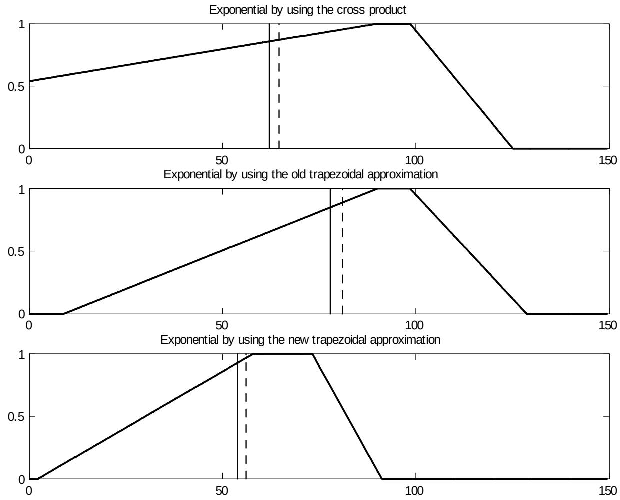

Figure 5
The exponential of a fuzzy number

Significant difference can be observed between the different results in this case. Indeed, the iterative use of the product operations leads to different result even after defuzzification. This problem can be avoided by considering and examining all the operations in all the practical problems considered and taking them into account there.

This figure suggests us also that we should be careful with the use of the cross product in the construction of an exponential, since in some cases the result given by this operation is negative, which can never be possible. The figure suggests that probably the best behavior is that of the new trapezoidal approximation.

Also, we observe that after defuzzification (by centroid method) the result of the cross product is usually smaller than that of the new trapezoidal approximation, which is smaller at its turn than the old trapezoidal approximation of the product however the results are not very different.

Conclusions and further research

New approaches for the multiplication of fuzzy numbers have been discussed both from theoretical and practical point of view. As a conclusion of this research we can state that the theoretical properties of the cross-product and the new trapezoidal approximation motivate the usefulness of both methods. The most conservative method is the old trapezoidal approximation. So, there exist reasons for using all the above mentioned approaches and to take into account the results of all approaches in applications (e.g., in risk analysis).

From the computational point of view, let us remark that the old trapezoidal approximation and the cross product can be computed in an easy way taking into account only the endpoints of the core and support of the trapezoidal operands and the extension principle may be avoided in these cases. In the case of the new trapezoidal approximation, since the implementation of this operation involves the use of the extension principle and then numerical integration on the side-functions of the fuzzy numbers, the computational complexity is high. This makes almost impossible to use the new trapezoidal approximation in iterative computations.

For further research we propose effective implementation of the new trapezoidal approximation, and the design of new computationally tractable methods for the approximation of the product based on the extension principle.

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