A Note on Integral Equivalence

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Abstract. In the note is given a new sufficient condition for a integral equivalence of a linear differential system and its nonlinear perturbation.

In [4] the following notion was defined:

Let two systems of differential equations

$$x' = F(t, x) \tag{1}$$

and

$$y' = G(t, x) \tag{2}$$

be given. Suppose that F and G are such that they guarantee the existence of solutions of (1) and (2), respectively, on the infinite interval $\langle 0, \infty)$.

Let $\Psi(t)$ be a positive continuous function on an interval $\langle t_0, \infty)$ and let $p > 0$. We shall say that two systems (1) and (2) are (Ψ, p)-integral equivalent on $\langle t_0, \infty)$ if for each solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that

$$\Psi^{-1}(t)\,|x(t) - y(t)| \in L_p(t_0, \infty) \tag{3}$$

and conversely, for each solution $y(t)$ of (2) there exists a solution x(t) of (1) such that (3) holds.

By restricted (Ψ, p)-integral equivalence between (1) and (2) we shall mean that relation (3) is satisfied for some subsets of solutions of (1) and (2), e.g. for the bounded solutions.

In this paper we shall use a topological method of Wazewski to discuss this problem. Now we shall define some notions and give preliminary results which will be needed in the sequel.

Hypothesis H. The system

$$\dot{x} = F(t, x), \tag{4}$$

where $x := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $F(t, x) := \begin{pmatrix} F_1(t, x_1, \cdots, x_n) \\ \vdots \\ F_n(t, x_1, \cdots, x_n) \end{pmatrix}$ and $(t, x) \in \Omega$,

satisfies the hypothesis H if

i) the real-valued functions $F_i(t, x)$, $i = 1, \dots, n$ of the real variables $t, x_1, \cdots, x_n$ are continuous in the set $\Omega \subset R^{n+1}$,

ii) through every point $P_0 = (t_0, x_0) \in \Omega$ passes only one integral curve $x(t, P_0)$ of the system (4).

Let ω and Ω be open sets of R^{n+1} with $\omega \subset \Omega$ and let us denote by $B(\omega, \Omega)$ the boundary of ω in Ω. Let $P_0 = (t_0, x_0) \in \Omega$. Denote $I(t, P_0) := (t, x(t, P_0))$ where $x(t, P_0)$ is the integral curve of the system (1) passing through the point P_0.

Let $(\alpha(P_0), \beta(P_0))$ be the maximal open interval in which the integral curve passing through P_0 exist. We shall write $I(\Delta, P_0) := \{(t, x(t, P_0)) \mid t \in \Delta\}$

for every $\Delta \subset (\alpha(P_0), \beta(P_0))$.

The point $P_0 = (t_0, x_0) \in B(\omega, \Omega)$ is a point of egress from ω (with respect to the system (4) and the set Ω) if there exists $\delta > 0$ such that $I([t_0 - \delta, t_0), P_0) \subset \omega$; P_0 is a point of strict egress from ω if P_0 is a point of egress and if there exists $\delta > 0$ such that $I((t_0, t_0 + \delta], P_0) \subset \Omega \setminus \overline{\omega}$. The set of all points of egress (strict egress) is denoted by S (S^*).

If *A* and *B* are any two sets of a topological space with $A \subset B$ and if $\pi : B \to A$ is a continuous mapping from *B* into *A* such that $\pi(P) = P$ for every $P \in A$, then π is a retraction from *B* into *A*, and *A* is a retract of *B*.

Wazewski's first theorem (T. Wazewski [6]). Suppose that the system (4) and the open sets $\omega \subset \Omega \subset R^{n+1}$ satisfy the following hypotheses:

i) hypothesis H

ii) $S = S^*$

iii) there exists a nonempty set $Z \subset \omega \cup S$ such that $Z \cap S$ is a retract of *S*, but it is not a retract of *Z*.

Then there exists at least one point $P_0 = (t_0, x_0) \in Z - S$ such that $I(t, P_0) \subset \omega$ for every *t*, $t_0 \le t < \beta(P_0)$.

Let $g = g(t,x) = g(t, x_1, \cdots, x_n) \in C^1(\Omega, R)$ i.e. let g be a real-valued function defined on $\Omega \subset R^{n+1}$, first partial derivatives of which exists and are continuons on Ω.

Let $P_0 = (t_0, x_0) \in \Omega$ and let *x*(*t*) be the integral curve of the system (4) passing through the point P_0. We set $\Phi(t) := g(t, x(t))$.

The derivative of *g*(*t*,*x*) at the point $P_0 = (t_0, x_0)$ with respect to the system (4) is by definition $\dot{\Phi}(t)$ and is denoted by $[D_{(4)} g(P)]_{P_0}$.

Let $l^i(t,x)$ and $m^j(t,x)$ $(i = 1, \cdots, p;\ j = 1, \cdots, q)$ be real-valued functions belonging to C^1 on an open set $\Omega \subset R^{n+1}$.

Let

$$\omega := \{P \in \Omega \mid l^i(P) < 0,\ i = 1, \cdots, p;\ m^j(P) < 0,\ j = 1, \cdots, q\},$$

$$L^i := \{P \in \Omega \mid l^i(P) = 0,\ l^k(P) \le 0,\ k = 1, \cdots, p\ ,\ m^j(P) \le 0,\ j = 1, \cdots, q\}$$

$$M^j := \{P \in \Omega \mid l^i(P) \le 0,\ i = 1, \cdots, p,\ \ m^j(P) = 0,\ m^k(P) \le 0,\ k = 1, \cdots, q\}$$

The set ω is called a regular polyfacial set, if

$$[D_{(4)} l^i(P)]_{P \in L^i} > 0,\ i = 1, \cdots, p \text{ and } [D_{(4)} m^j(P)]_{P \in M^j} < 0,\ j = 1, \cdots, q.$$

Wazewski's second theorem (T. Wazewski [6]). Let the system (4) satisfies the hypothesis H on a open set $\Omega \subset R^{n+1}$. Let $\omega \subset \Omega$ be a regular polyfacial set.

Then $S = S^* = \bigcup_{i=1}^{p} L^i \setminus \bigcup_{j=1}^{q} M^j$.

Lemma 1 (A. Haščák [2]). Let $g \geq 0$ be a continuous function on $0 \leq t < \infty$ and such that $\int_0^\infty s^{\frac{1}{p}} g(s)\,ds < \infty,\ \ p \geq 1$.

Then $\int_t^\infty g(s)\,ds \in L_{p'}(0,\infty),\ \ p' \geq p$.

Next we will consider special systems

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + f_i(t, x_1, \cdots, x_n) \tag{5}$$

and

$$\dot{y}_i = \sum_{j=1}^{n} a_{ij} y_j \ . \tag{6}$$

We will always suppose that the systems (5) and (6) satisfy the hypothesis H in $[T,\infty)\times\Gamma$, where $\Gamma \subset R^n$ is an open set. $|\cdot|$ denotes any convenient matrix (vector) norm.

Theorem 1 Suppose $y = y(t)$, $t \geq T$ is a given solution of (6) and there exist an $\varepsilon > 0$ and a $t_0 \geq T$ such that

$$w_{\varepsilon,t_0,y} := \bigcup_{t \geq t_0} \{t\} \times \{x \in R^n \mid \ \ |x - y(t)| < \varepsilon\} \subset \Omega = (T,\infty)\times\Gamma .$$

If there exist continuous functions $h_j(t),\ j = 1,\cdots,n$ such that

$$|f_j(t,x)| \leq h_j(t), \quad (t,x) \in w_{\varepsilon,t_0,y}$$

and if

$$\int_T^\infty \left\{ \int_t^\infty h_k(\tau) e^{\int_\tau^t a_{kk}(s)ds} d\tau \right\}^p dt < \infty, \quad k = 1,\cdots,n \ , \tag{7}$$

$$\int_T^\infty \left\{ \int_t^\infty |a_{ij}(\tau)| e^{\int_\tau^t a_{ii}(s)ds} d\tau \right\}^p dt < \infty, \quad i \neq j \ . \tag{8}$$

Then there exists $t_1 > t_0$ and a solution $x(t)$ of (1) defined on $< t_1, \infty)$ such that $| x(t) - y(t) | \in L_p(t_1, \infty)$.

Proof. Let $\Phi_i(t) \in L_p(t_1, \infty)$ for $1 \leq p < \infty, \quad i = 1, \cdots, n$ be such that

i) $\Phi_i(t) > 0, \ t \geq 0, \ i = 1, \cdots, n$.

ii) $\Phi_i(t)$ are differentiable and $\lim_{t \to \infty} \Phi_i(t) = 0, \ i = 1, \cdots, n$.

Thus there is $t_1 \in < T, \infty)$ such that $\sum_{i=1}^{n} \Phi_i(t) < \varepsilon < 1, \ t \geq t_1$.

Now we define the set ω by the formula

$\omega := \{P \in \Omega \,|\, | x_i - y_i(t) | < \Phi_i(t), \ t \geq t_1, \ i = 1, \cdots, n\}$.

If we put $l^i(P) := | x_i - y_i(t) |^2 - \Phi_i^2(t), \ i = 1, \cdots, n$ and $m^1(P) := t_1 - t$, then

$\omega = \{P \in \Omega \,|\, l^i(P) < 0, \ i = 1, \cdots, n$ and $m^1(P) < 0\}$.

Further let us define the sets $L^i, \ i = 1, \cdots, n$ and M^1 by the formulas

$$L^i := \{P \in \Omega \,|\, | x_i - y_i(t) | = \Phi_i(t), \ | x_j - y_j(t) | \leq \Phi_j(t), \ j = 1, \cdots, p, t \geq t_1\}$$
$$M^1 := \{P \in \Omega \,|\, | x_i - y_i(t) | \leq \Phi_i(t), \ t = t_1\}.$$

For the solution $x(t), t \geq t_1$ of (5) we have

$$\frac{1}{2} \dot{l}^i\big(t, x(t)\big) = | x_i(t) - y_i(t) |^2 \ a_{ii}(t) + \sum_{j \neq i}^{n} a_{ij}(t)\big(x_i(t) - y_i(t)\big)\big(x_j(t) - y_j(t)\big) +$$
$$+ f_i(t, x)\big(x_i(t) - y_i(t)\big) - \Phi_i(t).\dot{\Phi}_i(t), \quad i = 1, \cdots n$$

and thus $\frac{1}{2}[D_{(5)} l^i(P)]_{P \in L^i} \geq | x_i - y_i(t) |^2 \ a_{ii}(t) -$

$$- \sum_{j \neq i} | a_{ij}(t) | . | x_i - y_i(t) | . | x_j - y_j(t) | - | f_i(t, x) | . | x_i - y_i(t) | -$$

$$- \Phi_i(t).\dot{\Phi}_i(t) \geq \Phi_i^2(t).a_{ii}(t) - \sum_{j\neq i} | a_{ij}(t) | \Phi_i(t).\Phi_j(t) -$$

$$- | f_i(t,x) | \Phi_i(t) - \Phi_i(t).\dot{\Phi}_i(t) = \Phi_i(t)[a_{ii}(t).\Phi_i(t) - \sum_{j\neq i} | a_{ij}(t) | \Phi_j(t) -$$

$$- | f_i(t,x) | - \dot{\Phi}_i(t)] \geq \Phi_i(t)[\Phi_i(t).a_{ii}(t) - \dot{\Phi}_i(t) - \gamma(t)],$$

where $\gamma(t) := h_i(t) + \sum_{j\neq i} | a_{ij}(t) |$.

In order to be $[D_{(5)} l^i(P)] > 0,\ i = 1,\cdots,n$ it is sufficient to choose $\Phi_i(t),\ i = 1,\cdots,n$ such that $-\dot{\Phi}_i(t) + a_{ii}(t)\Phi_i(t) - \gamma(t) > 0$.

The problem is to find a solution $z(t)$ of

$$\dot{z}(t) < a_{ii}(t)z(t) - \gamma(t) \tag{9}$$

such that $z(t) > 0,\ z(t) \in L_p(0,\infty)$ and $\lim_{t\to\infty} z(t) = 0$.

The function $u(t) := \int_t^\infty h_i(\tau).e^{\int_\tau^t a_{ii}(s)ds} d\tau$ is a solution of the equation $\dot{u}(t) = a_{ii}(t)u(t) - h_i(t)$ such that $u(t) > 0,\ t \geq t_1,\ \lim_{t\to\infty} u(t) = 0$ and (by (7)) $u(t) \in L_p(t_1,\infty)$.

Moreover the function

$$v_j(t) := \int_t^\infty | a_{ij}(t) | e^{\int_\tau^t a_{ii}(s)ds} d\tau,\ j \in \{1,\cdots,i-1, i+1,\cdots,n\}$$

is a solution of the equation

$\dot{v}_j(t) = a_{ii}(t).v_i(t) - | a_{ij}(t) |$, $j \in \{1,\cdots,i-1, i+1,\cdots,n\}$ such that $v_j(t) > 0,\ t \geq t_1,\ \lim_{t\to\infty} v_j(t) = 0$ and by (8) $v_j(t) \in L_p(t_1,\infty)$.

Thus the function $w(t) := u(t) + \sum_{j\neq i} v_i(t),\quad t \geq t_1$ is solution of the equation

$$\dot{w}(t) = a_{ii}(t)w(t) - h_i(t) - \sum_{j\neq i} | a_{ij}(t) | \tag{10}$$

such that $w(t) > 0,\ \ t \ge t_1,\ \ \lim_{t\to\infty} w(t) = 0$.

Further, by Holder's inequality we have $w(t) \in L_p(t_1, \infty)$.

Moreover from (10) we have

$$2\dot{w}(t) = 2a_{ii}(t)w(t) - 2\left(h_i(t) + \sum_{j\neq i} |\, a_{ij}(t)\,|\right) <$$

$$< 2a_{ij}(t)w(t) - h_i(t) - \sum_{j\neq i} |\, a_{ij}(t)\,|,$$

i.e. the function $z(t) = 2w(t)$ is a solution of (9) with required property.

Now (since $[D_{(5)}, m^1(P)]_{P\in M^1} = -1$) it follows from Wazewski's Second Theorem, that ω is a polyfacial set and $S = S^* = \left[\bigcup_{i=1}^{n} L^i\right] - M^1$.

Let *Z* be the set

$$Z := \{(t,x) \in R^{n+1} \mid t = \tau > t_1, |\, x_j - y_j(t) | \le \Phi_j(t),\ \ j = 1,\cdots,n\}.$$

Then $S \cap Z = \left[\bigcup_{i=1}^{n} L^i \cap Z\right] - M^1$, where

$$L^i \cap Z =$$

$$= \{(t,x) \mid\ t = \tau,\ |\, x_i - y_i(\tau)\,| = \Phi_i(\tau), |\, x_j - y_j(\tau)\,| \le \Phi_j(\tau),\ j = 1,\cdots,n\}$$

Thus $Z = B_1 \times \cdots \times B_n$, where B_j, are the balls in R^1, and

$S \cap Z = \bigcup_{j=1}^{n} B_1 \times \cdots \times S_j \times B_{j+1} \times \cdots \times B_n$, where S_j is a boundary of B_j in R.

Also, modulo homeomorphisms, $Z = B^n$ (a ball in R^n) and $Z \cap S$ is the boundary of B^n in R^n. Thus $Z \cap S$ is not a retract of Z. On the other hand, since the function $\pi : S \to S \cap Z$ defined by

$$\pi : (t, x_1, \cdots, x_n) \to$$

$$\to \left(\tau, y_1(\tau) + [x_1 - y_1(t)]\frac{\Phi_1(\tau)}{\Phi_1(t)}, \cdots, \; y_n(\tau) + [x_n - y_n(t)]\frac{\Phi_n(\tau)}{\Phi_n(t)} \right)$$

is a retraction, i.e. $Z \cap S$ is a retract of S.

By Wazewski's First Theorem there is at least one point $P_0 \in (\tau, x_0) \in Z \setminus S$ such that $\left(t, x(t, P_0)\right) = I(t, P_0) \subset \omega, \; t \geq \tau$.

It must be that $\beta(P_0) = \infty$ because otherwise

$\{I(t, P_0) \mid \quad \tau \leq t < \beta(P_0)\} \cap [\Omega - \omega] \neq \emptyset$, which is not possible.

Consequently, $x(t, P_0) = \left(x_1(t, P_0), \cdots, x_n(t, P_0)\right)$ is defined in $[\tau, \infty)$, and such that $| x_i(t) - y_i(t) | \in L_p(\tau, \infty), \; i = 1, \cdots, n$.

Moreover, if $| x | := \max\left(| x_1 |, \cdots, | x_n |\right)$ for $x \in R^n$, then

$$\int_\tau^\infty | x(t) - y(t) |^p \, dt \leq \int_\tau^\infty \sum_{i=1}^n | x_i(t) - y_i(t) |^p \, dt < \infty .$$

The proof of the theorem is complete.

In the next $U(t)$ will denote a fundamental matrix of (6).

Corollary 1. Suppose that the following conditions hold:

i) all solutions of (6) are defined and bounded on $[T, \infty)$,

ii) in the system (5), $f(t, x)$ is defined on $[T, \infty) \times R^n$,

iii) for every constant M and some $t_0 > T$, there exists a continuous real-valued function $h_M(t)$ such that

$$\int_t^\infty h_M(s) | U^{-1}(s) | \, ds \in L_p(t_0, \infty)$$

and $| f(t, x) | \leq h_M(t)$ for all (t, x) with $t > t_0, \; | x | \leq M$.

Then for every solution $y(t)$ of (6) there is a solution $x(t)$ of (5) such that

$| x(t) - y(t) | \in L_p(t_1, \infty)$.

Proof. Using the transformations $x(t) = U(t).z(t), \quad y(t) = U(t).v(t)$ in the systems (5) and (6), we get

$$\dot{z}(t) = U^{-1}(t) f\big(t, U(t)z(t)\big) =: g(t,z), \tag{11}$$

and

$$\dot{v}(t) = 0. \tag{12}$$

Further, by the hypothesis i) there exists a constant *K* such that $|U(t)| \le K$, for $t \ge t_0 \ge T$. Clearly $|x(t) - y(t)| \le K\,|z(t) - v(t)|$.

Thus to prove that for every solution $y(t)$ of (6), there is a solution $x(t)$ of (5) such that $|x(t) - y(t)| \in L_p(t_0, \infty)$, it is enough to prove $|z(t) - v(t)| \in L_p(t_0, \infty)$.

We have $|g(t,z)| \le |U^{-1}(t)|.|f(t, U(t)z)| \le |U^{-1}(t)|\, h_{\widetilde{M}}(t), \ t \ge t_0$

for $|z| \le M$, where $\widetilde{M} = KM$ and by hypothesis

$$\int_t^\infty |U^{-1}(s)|\, h_M(s)\, ds \in L_p(t_0, \infty)$$

holds. Now the existence of solution $z(t)$ of (11) with required property follows from Theorem 1 applied to the systems (11) and (12).

From Corollary 1 and Lemma 1 we have:

Corollary 2. Assume that the assumptions of Corollary 1 are satisfied except ii) which is substituted by the condition

iv) for every constant *M* and some $t_0 > T$, there exists a continuous real-valued function $h_M(t)$ such that $\int_t^\infty |s^{\frac{1}{p}} h_M(s)|U^{-1}(s)|ds < \infty$ and $|f(t,x)| \le h_M(t)$, for all (t,x) with $t > t_0, \ |x| \le M$.

Then the conclusion of Corollary 1 holds true.

Theorem 2. Assume that the assumptions of Corollary 2 are satisfied. Then for every bounded solution $x(t)$ of (5), there exists a solution $y(t)$ of (6) such that $|x(t) - y(t)| \in L_p(t_0, \infty)$.

Proof. Let $x(t)$ be a solution of (5) such that $|x(t)| \le M, t \ge t_0$. Let us consider the solution $\tilde{y}(t)$ of (6) definited by the integral equation $x(t) = \tilde{y}(t) + \int_{t_0}^{t} U(t)U^{-1}(s) f\left(s, x(s)\right) ds$.

Clearly, $|U^{-1}(s) f\left(s, x(s)\right)| \le |U^{-1}(s)| h_M(s), \quad s \ge t_0$ and by iv) of Corollary 2. we have $\int_{t_0}^{\infty} |U^{-1}(s) f\left(s, x(s)\right)| ds < \infty$.

Therefore,

$$x(t) = \tilde{y}(t) + U(t)\int_{t_0}^{\infty} U^{-1}(s) f\left(s, x(s)\right) ds + \int_{\infty}^{t} U^{-1}(s) f\left(s, x(s)\right) ds =$$

$= y(t) + \int_{\infty}^{t} U^{-1}(s) f\left(s, x(s)\right) ds$, where $y(t)$ is a solution of (6).

From the assumption iv) of Corollary 2 and Lemma 1 it follows that $|x(t) - y(t)| \in L_p(t_0, \infty)$.

The proof of Theorem 2 is complete.

Theorem 3 Suppose that the assumptions i) and iii) of Corollary 2 are satisfied and further suppose that

v) for every constant $M > 0$ and some $t_0 > T$ there exists a continuous real-valued function $h_M(t)$ such that $\int_{t_0}^{\infty} t^{\frac{1}{p}} h_M(t).e^{\int_{t}^{t_0} \sum_{j=1}^{n} a_{jj}(s)\, ds} dt < \infty$ and $|f(t,x)| \le h_M(t)$, for all (t,x) with $t > t_0$, $|x| \le M$.

Then the sets of bounded solutions of (5) and of (6) are (1, *p*)-integral equivalent.

Proof. From the Jacobi-Liouville formula $\det U(t) = \det U(t_0) e^{\int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s)\, ds}$,

it follows that $|[\det U(t)]^{-1}| = |[\det U(t_0)]^{-1}| e^{\int_{t}^{t_0} \sum_{j=1}^{n} a_{jj}(s)\, ds}$.

Since $U^{-1}(t) = [\det U(t)]^{-1} adjU(t)$ and hypothesis i) implies that adj $U(t)$ is bounded, it is clear that $|U^{-1}(t)| = |[\det U(t)]^{-1}|.|adjU(t)| =$

$= |[\det U(t)]^{-1}|.|adjU(t)| e^{\int_t^{t_0} \sum_{j=1}^{n} a_{jj}(s)\,ds} \leq K.e^{\int_t^{t_0} \sum_{j=1}^{n} a_{jj}(s)\,ds}$, for some constant K. Therefore $\int_{t_0}^{\infty} t^{\frac{1}{p}} h_M(t) |U^{-1}(t)|\,dt < \infty$.

Now the conclusion of Theorem 3 follows from Corollary 2 and Theorem 2.

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