Graceful Labelling of Special Halin Graph

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Abstract: We deal with the problem of graceful labelling the vertice of a special class of a modification of Halin graph with the tree $S_2 + S_n$*, where* S_n *is star. MSC: 05C78, 05C05*

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1 Introduction

Graph labellings is an active area of research in Graph Theory which has mainly evolved through its many applications in Coding Theory, Communication Networks, Mobile Telecomminication Systems, Optimal Circuits Layouts or Graph Decompositions problems, no name just a few of them.

By a graph we mean a finite graph without loops and multiple edges. Terms and notations not defined bellow follow that used in [1], [2].

A vertex labelling (or valuation) of a graph $G = (V, E)$ is an assignment f of labels to the vertices of $V(G)$ that induces for each edge $uv \in E(G)$ a label depending on the vertex labels $f(u)$ and $f(v)$. Let G be a graph with q edges and let $f : V(G) \to \{0,1,\ldots,q = |E(G)|\}$ be an injection. A vertex labelling f is called a graceful labelling if for each edge uv the absolute value $|f(u) - f(v)| = w(u,v)$ is assigned and the resulting edge labels are mutually distinct. The value $w(u,v)$ is called the edge-weight of the edge uv. A graph possessing a graceful labelling is called a graceful graph.

The graceful labelling is one of the most popular king of graph labelling.

Let $G(k,l)$ be a planar graph whose edgeset E can be decomposit into two disjoint subsets T and C, $E = T \cup C, T \cap C = \emptyset$, where the subgraph of $G(k,l)$ induced on T is a tree with one vertex u of degree k, one vertex v of degree l, u and v being adjacent, and the remaining $k+l-1$ vertices of degree one and the subgraph induced on C is a cycle of length $k+l-1$ passing through all vertices of $G(k,l)$ except of including u and v. Note that in the case $k \geq l \geq 3$, the $G(k,l)$ is a special Halin graph [3].

Figure 1 displayes the graph $G(4,5)$ where the cycle C is depicted with thick lines.

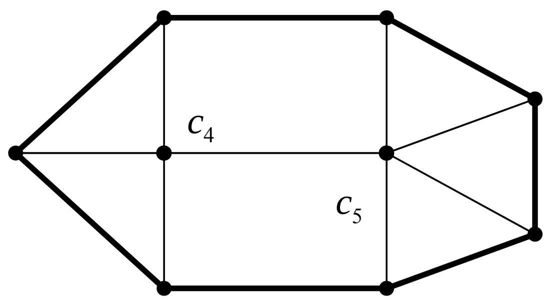

Figure 1
Halin graph G(4, 5)

The main purpose of this note is to show that the graph $G(2,n)$, $n \geq 3$ are graceful.

2 Graceful Labelling of $G(2,n)$

Let $n \geq 3$ and $G(2,n)$ be the graph with the vertex set $V = \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ and the edge set

$$E = \bigcup_{i=1}^{n} \{x_i, x_{i+1}\} \cup \bigcup_{j=1}^{n-1} \{x_0, x_j\} \cup \{x_0, x_{n+1}\} \cup \{x_1, x_n\}$$ [see Figure 2].

Then $|V| = n+2$ and $|E| = 2n+1$.

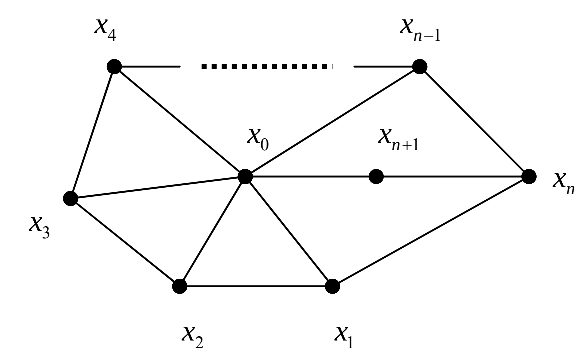

Figure 2
Graph G(2, n)

On Figure 3 are graceful labellings of $G(2,3)$ and $G(2,4)$.

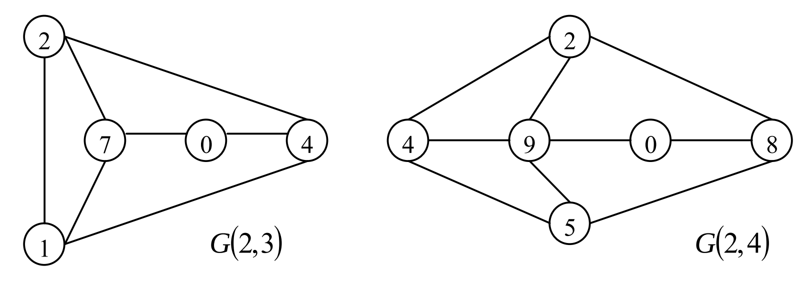

Figure 3
Graceful labelling of G(2, 3) and G(2, 4)

We define for $n \geq 5$ the vertex labelling $f: V \rightarrow \{0,1,2,...,2n+1\}$ in the following way:

$$f(x_{n+1}) = 0, \; f(x_n) = 2n, \; f(x_0) = 2n+1 \text{ and}$$

<u>Case 1.</u> n is odd, $n \geq 5$.

Then $f(x_{n-1}) = 2n-1$ and $f(x_i) = \begin{cases} i+1 & if \;\; i \in \{1,3,...,n-2\} \\ 2n-i & if \;\; i \in \{2,4,...,n-3\} \end{cases}$

It is easy to see that

$w(x_i, x_{i+1}) = 2(n-i-1)$ for $i \in \{1,2,...,n-3\}$, $w(x_0, x_{n-1}) = 2$, $w(x_1, x_n) = 2n-2$, $w(x_n, x_{n+1}) = 2n$ (all even weights of edges)

and $w(x_0, x_i) = 2n-i$ for $i \in \{1,3,...,n-2\}$, $w(x_0, x_i) = i+1$ for $i \in \{2,4,...,n-3\}$, $w(x_{n-1}, x_n) = 1$, $w(x_{n-2}, x_{n-1}) = n$, $w(x_0, x_{n+1}) = 2n+1$ (all odd weights of edges).

Case 2. n is even, $n \geq 6$.

Then $f(x_1) = 2$, $f(x_2) = 4$, $f(x_3) = 5$ and

$$f(x_i) = \begin{cases} n+i & if \quad i \in \{4,6,...,n-2\} \\ n-i+5 & if \quad i \in \{5,7,...,n-1\} \end{cases}$$

We have $w(x_i, x_{i+1}) = 2i-4$ for $i \in \{4,5,...,n-1\}$, $w(x_1, x_2) = 2$, $w(x_0, x_3) = 2n-4$, $w(x_1, x_n) = 2n-2$, $w(x_n, x_{n+1}) = 2n$ (all even weights of edges)

and $w(x_0, x_i) = n-i+1$ for $i \in \{4,6,...,n-2\}$, $w(x_0, x_i) = n+i-4$ for $i \in \{5,7,...,n-1\}$, $w(x_2, x_3) = 1$, $w(x_3, x_4) = n-1$, $w(x_0, x_1) = 2n-1$, $w(x_0, x_2) = 2n-3$, $w(x_0, x_{n+1}) = 2n+1$ (all odd weights of edges).

Clearly this vertex labelling is graceful. Thus we have proved.

<u>Theorem</u>: Graph $G(2,n)$, $n \geq 3$ ***is graceful.***

Conclusion and open problems

In this paper we presented an infinity family of graceful graphs. A question arise what about graphs $G(m,n)$, $m,n \geq 3$? A special case $G(n,n)$, $n \geq 3$ can be particulary interesting. We are very little results:

$G(3,n)$ are graceful for $n \leq 6$ and $G(n,n)$ are graceful for $n \leq 5$. For example on Figure 4 are graceful labellings of $G(3,6)$ and $G(5,5)$.

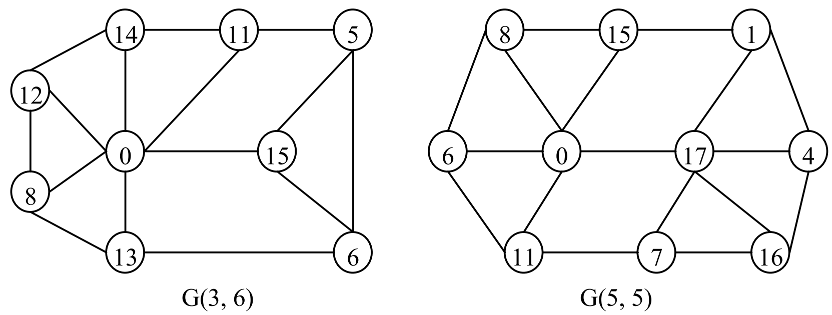

Figure 4

Graceful labelling of G(3, 6) and G(5, 5)

References

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