Bivariate Spline Approximation of Fuzzy Functions and Its Use in Digital Terrain Modeling

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Abstract. The usefullness of fuzzy input fuzzy output functions and their interpolation/approximation by different operators in fuzzy control motivate their deeper theoretical study. We obtain some properties of bivariate fuzzy B-spline series such as variation and uncertainty diminishing property. We also discuss the use of such operators in digital terrain modelling.

1 Introduction

Fuzzy numbers can model linguistic expressions of the type "around", "early morning", "small", "big", etc. Due to the modelling ability of fuzzy numbers their study plays actually a central role in fuzzy sets theory (fuzzy mathematics). The usefulness of fuzzy arithmetic in Geography and Geology is shown recently in several works (see [2], [4]). Fuzzy-number-valued functions (in the present paper we call them fuzzy functions) are a natural way to model uncertain temporal or spatial dependency with respect to some crisp real variables. This remark suggests us their possible usefulness in Geographical and Geological applications where temporal or spatial dependency is subject of non-probabilistic uncertainty. In this paper we discuss bivariate fuzzy B-spline approximation and it's application in digital terrain modelling.

In [16], L. Zadeh proposed the problem of interpolating some fuzzy data. This problem was solved in [13] and [9]. Also, recently in [5] the authors give error estimate in polynomial and trigonometric approximation, and the convergence of fuzzy Lagrange interpolation polynomial is studied. In [9] the notion of interpolating fuzzy spline is introduced. In [1] and [3], complete and natural splines interpolating fuzzy data are considered. For approximation of fuzzy-number-valued functions these kinds of interpolating fuzzy splines are not very practical since their coefficients change sign at each knot, and it is difficult (or impossible) to prove approximation theorems (by lack of distributivity for operations with fuzzy numbers). Fuzzy B-spline series are introduced in [2] (there they are called fuzzy B-splines). Approximation properties of fuzzy B-spline series are not studied in [2], however their applications motivate an accurate study of them. Recently, in [11], having as starting point fuzzy interpolation, consistent fuzzy surfaces are constructed from fuzzy data. In [2] the bivariate fuzzy B-spline series are used in digital terrain modeling. We further investigate in the present paper bivariate fuzzy B-spline series, both from the theoretical and practical point of view.

Fuzzy B-spline series are studied from the theoretical point of view in the recent paper [6]. Error bounds for approximation of a continuous fuzzy function by fuzzy B-spline series are obtained in terms of the modulus of continuity. Particularly simple error bounds are obtained for approximation by fuzzy B-spline series of Schoenberg type.

After a preliminary section, we recall in Section 3 the fuzzy B-spline series and fuzzy splines of Schoenberg-type which are spline extensions of Bernstein polynomials and we remind some properties such as continuity, variation and uncertainty diminishing property. Section 4 is concerned with the approximation of a bivariate fuzzy function by some bivariate fuzzy B-spline series. Jackson-type estimates and particularly simple error bounds are obtained by using the modulus of continuity. Some conclusions and further research topics conclude the paper.

2 Preliminaries

Let $\mathbf{R}_F$ denote the space of fuzzy numbers.

For $0 < \alpha \le 1$ and $u \in \mathbf{R}_F$ let $[u]^\alpha = \{x \in \mathbf{R}; u(x) \ge \alpha\}$ and $[u]^0 = \overline{\{x \in \mathbf{R}; u(x) > 0\}}$. Then it is well known that for each $\alpha \in [0,1]$, $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ is a bounded closed interval ($\underline{u}^\alpha, \overline{u}^\alpha$ denote the endpoints of the $\alpha-$ level set). For $u, v \in \mathbf{R}_F$ and $\lambda \in \mathbf{R}$, we have the sum $u + v$ and the product

$\lambda \cdot u$ defined by $[u+v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$, $[\lambda \cdot u]^{\alpha} = \lambda [u]^{\alpha}$, $\forall \alpha \in [0,1]$, where $[u]^{\alpha} + [v]^{\alpha}$ means the usual addition of two intervals (as subsets of $\mathbf{R}$) and $\lambda [u]^{\alpha}$ means the usual product between a scalar and a subset of $\mathbf{R}$. A fuzzy number $u \in \mathbf{R}_F$ is said to be positive if $\underline{u}^1 \geq 0$, strict positive if $\underline{u}^1 > 0$, negative if $\overline{u}^1 \leq 0$ and strict negative if $\overline{u}^1 < 0$. We say that u and v have the same sign if they are both positive or both negative. If u is positive (negative) then $-u = (-1) \cdot u$ is negative (positive).

A special class of fuzzy numbers is the class of triangular fuzzy numbers. Given $a \leq b \leq c$, $a,b,c \in \mathbf{R}$, the triangular fuzzy number $u = (a,b,c)$ determined by a,b,c is given such that $\underline{u}^{\alpha} = a + (b-a)\alpha$ and $\overline{u}^{\alpha} = c - (c-b)\alpha$, for all $\alpha \in [0,1]$. Then $\underline{u}^0 = a$, $\underline{u}^1 = \overline{u}^1 = b$ and $\overline{u}^0 = c$.

Define $D : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}_+ \cup \{0\}$ by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max \left\{ \left| \underline{u}^{\alpha} - \underline{v}^{\alpha} \right|, \left| \overline{u}^{\alpha} - \overline{v}^{\alpha} \right| \right\}.$$

The following properties are known:

$D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u,v,w \in \mathbf{R}_F$

$D(k \cdot u, k \cdot v) = |k| D(u,v)$, $\forall u,v \in \mathbf{R}_F$, $\forall k \in \mathbf{R}_F$;

and $(\mathbf{R}_F, D)$ is a complete metric space.

Also, the following properties are known.

(i) If we denote $\tilde{0} = \chi_{\{0\}}$ then $\tilde{0} \in \mathbf{R}_F$ is neutral element with respect to $+$, i.e. $u + \tilde{0} = \tilde{0} + u = u$, for all $u \in \mathbf{R}_F$.

(ii) With respect to $\tilde{0}$, none of $u \in \mathbf{R}_F - \mathbf{R}$ has opposite in $\mathbf{R}_F$ (with respect to $+$).

(iii) For any $a,b \in \mathbf{R}$ with $a,b \geq 0$ or $a,b \leq 0$, and any $u \in \mathbf{R}_F$, we have

$(a+b) \cdot u = a \cdot u + b \cdot u$. For general $a,b \in \mathbf{R}$, the above property does not hold.

(iv) For any $\lambda \in \mathbf{R}$ and any $u,v \in \mathbf{R}_F$, we have

$\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v$.

(v) For any $\lambda, \mu \in \mathbf{R}$ and any $u \in \mathbf{R}_F$, we have

$\lambda \cdot (\mu \cdot u) = (\lambda \cdot \mu) \cdot u$.

(vi) If we denote $\|u\|_F = D(u, \tilde{0})$, $\forall u \in \mathbf{R}_F$, then $\|\cdot\|_F$ has the properties of an usual norm on $\mathbf{R}_F$, i.e. $\|u\|_F = 0$ iff. $u = \tilde{0}$, $\|\lambda \cdot u\|_F = |\lambda| \cdot \|u\|_F$ and $\|u+v\|_F \le \|u\|_F + \|v\|_F$, $\left| \|u\|_F - \|v\|_F \right| \le D(u,v)$.

The uniform distance between fuzzy-number-valued functions is defined by

$$D(f,g) = \sup\{D(f(x), g(x) \mid x \in [a,b]\}$$

for $f, g : [a,b] \to \mathbf{R}_F$.

For (X, d) any metric space and $f : X \to \mathbf{R}_F$ the function $\omega(f,\cdot): \mathbf{R}_+ \to \mathbf{R}$

$$\omega(f,\delta) = \sup\{D(f(x), f(y)) \mid x, y \in X, d(x,y) \le \delta\}$$

is called the modulus of continuity of the fuzzy function f.

The fuzzy splines as introduced by O. Kaleva in [9], are given below.

Definition 1 Let S_l be the family of all splines of order l with the knots t_i, $i = 0,1,...,n$. Then $fs(x) = \sum_{i=0}^{n} s_i(x) \cdot u_i$, where $s_i \in S_l$ is the crisp spline interpolating the data (x_i, f_j), $j = 0,1,...,n$, where $f_j = 1$ if $i = j$ and 0 otherwise, and $u_i \in \mathbf{R}_F$ are fuzzy constants is called a fuzzy spline.

3 Fuzzy B-Spline Series

Firstly, let us recall the definitions of the crisp B-splines. Let $t_0 \le t_1 \le .... \le t_r$ be points in $\mathbf{R}$, with $t_r \ne t_0$. The B-spline M is given by

$M(x) = M(x; t_0,...,t_r) = r[t_0,...,t_r](\cdot - x)_+^{r-1}$,

where $[t_0,...,t_r]f$ denotes the divided difference of f (see e.g. [12]).

The B-spline N is defined by

$$N(x; t_0,...,t_r) = \frac{1}{r}(t_r - t_0) M(x; t_0,...,t_r).$$

The fuzzy B-spline series are defined as follows (see [2]).

Let $A = [a,b]$ or $A = \mathbf{R}$. Let $T = (t_i)$ be a sequence of points in A called basic knots satisfying $t_i \le t_{i+1}$ and $t_i < t_{i+r}$, for any $t_i \in A$, $i = 0,...,n$ if $A = [a,b]$ and $i \in \mathbf{Z}$ if $A = \mathbf{R}$. If $A = [a,b]$ we need some auxiliary knots $t_{-r+1} \le ... \le t_0 = a$ and $b = t_{n+1} \le ... \le t_{n+r}$. To a given sequence of knots corresponds a sequence of crisp B-splines $N_j(x) = N(x; t_j,...,t_{j+r})$, for $j \in \Lambda$, where $\Lambda = \mathbf{Z}$ if $A = \mathbf{R}$ and $\Lambda = \{-r+1,...,n\}$ if $A = [a,b]$.

Definition 2 A fuzzy B-spline series on A ($A = \mathbf{R}$ or $A = [a,b]$) having knots in $T = (t_i)$, $i \in \Lambda$ is a function $S : A \to \mathbf{R}_F$, of the form

$$S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j,$$

where $c_j \in \mathbf{R}_F$.

First of all we recall some properties obtained in [6], such as continuity, the variation and uncertainty diminishing property of fuzzy B-spline series. Let us recall here two useful properties of crisp B-splines (see e.g. [11]).

$$N_j(x) \ge 0 \text{ for } x \in [t_j, t_{j+r}] \text{ and } N(x) = 0 \text{ for } x \notin [t_j, t_{j+r}]$$

The following identity holds

$$\sum_{j\in\Lambda} N_j(x) = 1.$$

Theorem 1 The fuzzy B-spline series $S(x)$ given in Definition fBs is continuous as function of x, the knots t_i and the coefficients c_j.

Next we recall the variation diminishing property. We say that a fuzzy-number-valued function changes sign in $[x_0, x_1]$ if $f(x_0)$ is negative (positive) and $f(x_1)$ is positive (negative). In digital terrain modeling this property can be interpreted as follows: The fuzzy B-spline approximation does not increase the spatial variability of the terrain (if interpolatory methods are used this can happen frequently).

Theorem 2 A fuzzy B-spline series $S(x) = \sum_{j\in\Lambda} N_j(x) \cdot c_j$ has the variation diminishing property, i.e. S changes its sign at most as many times as the sequence $(c_j)_{j\in\Lambda}$ changes it's sign.

The next property shows that fuzzy B-spline series can be useful in several application since they do not increase the uncertainty about the original data. For example, in digital terrain modeling this property shows us that the uncertainty on the altitude at each point is not overestimated, so the conservativism of the approximation is not excessive.

We denote by $len(u)$ the length of the support of the fuzzy number $u \in \mathbf{R}_F$, *i.e.* $len(u) = \overline{u}^0 - \underline{u}^0$.

Theorem 3 A fuzzy B-spline series $S(x) = \sum_{j \in \Lambda} N_j(x) \cdot c_j$ has the uncertainty diminishing property, i.e. the length of the support of $S(x)$ does not exceed the maximum length of the support of the fuzzy numbers c_j. (the length of the support of a fuzzy number can be interpreted as the uncertainty on it).

Remark 1 We observe that the splines defined by [kaleva] cannot be written as fuzzy B-spline series. Indeed, by Curry-Schoenberg theorem (see e.g. [11]), the crisp B-splines are a basis for the Schoenberg space of all splines. Let s_i be the splines in Definition splold. Then

$$s_i(x) = \sum_{j \in \Lambda} N_j(x) d_{ij},$$

with $d_{ij} \in \mathbf{R}$. Let fs be as in Definition splold. Then

$$fs(x) = \sum_{i=0}^{n} s_i(x) \cdot u_i = \sum_{i=0}^{n} \left(\sum_{j \in \Lambda} N_j(x) d_{ij} \right) \cdot u_i,$$

where $u_i \in \mathbf{R}_F$, $i = 0, ..., n.$ By the lack of the distributivity of the scalar multiplication with respect to the addition of fuzzy numbers, the two sums cannot be interchanged, because the splines s_i (and so also the coefficients d_{ij}) change their sign at each knot. Changing the order of the sums is possible if all d_{ij} have the same sign for $j \in \Lambda.$ The same remark is true for fuzzy splines defined in [1] and [3].

Remark 2 Let us observe that the fuzzy B-spline series $S(x) = \sum_{j \in \Lambda} N_j(x) \cdot c_j$ can be easily computed by using fuzzy arithmetic (i.e. addition of fuzzy numbers and multiplication of a fuzzy number by a crisp real).

Example 1 In what follows we give an example, and we compare our results with the examples in [1], [3] and [9]. In these examples we use triangular fuzzy numbers.

We construct the cubic fuzzy B-spline series approximating the triangular fuzzy data

$\xi_{-2} = 1,\ f(\xi_{-2}) = (-2,0,1),$
$\xi_{-1} = 1.1,\ f(\xi_{-1}) = (4,5,7),$
$\xi_{0} = 1.2,\ f(\xi_{0}) = (1,1,4),$
$\xi_{1} = 3,\ f(\xi_{1}) = (0,4,7),$
$\xi_{2} = 3.5,\ f(\xi_{2}) = (-3,0,2),$
$\xi_{3} = 4,\ f(\xi_{3}) = (0,1,2),$

with the knots $t_{-2} = 0.5$, $t_{-1} = 1.05$, $t_0 = 1.15$, $t_1 = 1.25$, $t_2 = 1.5$, $t_3 = 2.75$, $t_4 = 3.25$, $t_5 = 3.75$, $t_6 = 4.25$. The endpoints of the $0-$ level set and the $1-$ level set of the fuzzy B-spline series can be seen in Figure 1.

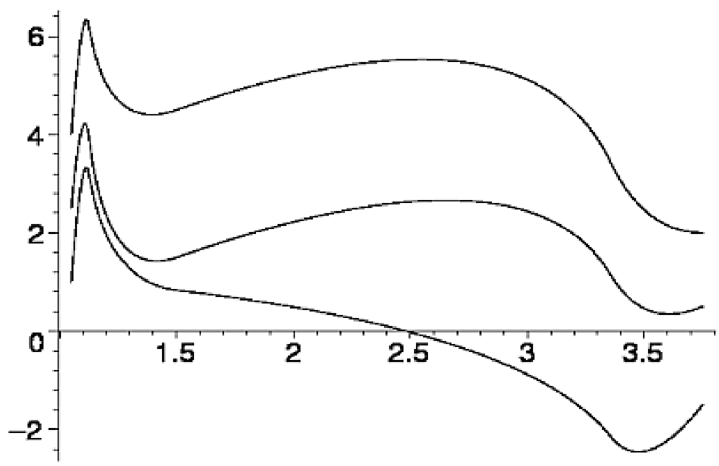

Figure 1
An example of a fuzzy B-spline

It is easy to observe that the uncertainty (i.e. the length of the 0 -level set) is smaller compared with the examples in [1], [3] and [kaleva] interpolating the same data.

4 Bivariate Fuzzy B-spline Series

The fuzzy B-spline series in Definition fBs can be extended easily to the bivariate case.

Definition 3 For a given rectangular grid of knots we consider

$$S(x,y)=\sum_{j\in\Lambda_1}\sum_{k\in\Lambda_2}N_{j,r_1}(x)N_{k,r_2}(y)\cdot c_{j.k},$$

where $N_{j,r_1}(x)$ is the crisp B-spline with respect to variable x, $N_{k,r_2}(y)$ is the crisp B-spline with respect to y and $c_{j.k}$ are fuzzy constants. Then $S(x,y)$ is called bivariate fuzzy B-spline series or fuzzy B-spline surface (see also [2], [11]).

The following remark shows us that the bivariate fuzzy B-spline series determine a so-called "consistent fuzzy surface" (see [11]).

Remark 3 Since

$$\underline{S(x,y)}^{\alpha}=\sum_{j\in\Lambda_1}\sum_{k\in\Lambda_2}N_{j,r_1}(x)N_{k,r_2}(y)\underline{c_{j,k}}^{\alpha}$$

and

$$\overline{S(x,y)}^{\alpha}=\sum_{j\in\Lambda_1}\sum_{k\in\Lambda_2}N_{j,r_1}(x)N_{k,r_2}(y)\overline{c_{j,k}}^{\alpha}$$

for any $\alpha\in[0,1]$, a fuzzy B-spline surface is a consistent fuzzy surface in the sense of [11], i.e.

(1) if $\alpha_1\le\alpha_2$

$$\underline{S(x,y)}^{\alpha_1}\le\underline{S(x,y)}^{\alpha_2}\le\overline{S(x,y)}^{\alpha_2}\le\overline{S(x,y)}^{\alpha_1},$$

i.e. surfaces with larger α -cut values are contained in surfaces with lower α -cut values, and

(2) $\underline{S(x,y)}^{\alpha}$ and $\overline{S(x,y)}^{\alpha}$ are crisp B-spline series, so they posses the underlying smoothness and continuity properties of the approximation method which is used.

We approximate a continuous bivariate fuzzy-number-valued function $f:[0,1]\times[0,1]\to\mathbf{R}_F$ by some fuzzy B-spline series $S:[0,1]\times[0,1]\to\mathbf{R}_F$. For this aim we consider the sequences of knots $0<t_1\le....\le t_n<1,$ auxiliary knots $t_{-r_1+1}\le....\le t_0=0,$ $1=t_{n+1}\le....\le t_{n+r_1}$ on and knots $0<t_1^{'}\le....\le t_m^{'}<1$ and auxiliary knots $t_{-r_2+1}^{'}\le....\le t_0^{'}=0,$ $1=t_{m+1}^{'}\le....\le t_{m+r_2}^{'}$. Let $N_{j,r_1}(x)$ denote the crisp B-splines with respect to the variable x having knot sequence t_j, $j=-r_1+1,...,n$ and $N_{k,r_2}(y)$ the crisp B-spline series with respect to variable y and knot sequence $t_k^{'}$, $k=-r_2+1,...,m.$ Let $(\xi_j,\eta_k)\in([0,1]\times[0,1])\cap(\text{supp}\,N_{j,r_1}\times\text{supp}\,N_{k,r_2})$ (where

$\operatorname{supp} N_{j,r} = \{x \in [0,1]:\ N_{jr}(x) \neq 0\})$.

Definition 4 The bivariate fuzzy B-spline series approximating a target function f are

$$S(f,x,y) = \sum_{j=-r_1+1}^{n} \sum_{k=-r_2+1}^{m} N_{j,r_1}(x) N_{k,r_2}(y) \cdot f(\xi_j, \eta_k).$$

We observe that everything is also valid for $f\ :[a,b]\times[c,d] \to \mathbf{R}_F$.

Approximation properties of bivariate fuzzy B-spline series are given in the following theorem.

Theorem 4 For $f\ :[0,1]\times[0,1] \to \mathbf{R}_F$ continuous we have:

$$D(f(x,y), S(f,x,y)) \le r_1 \cdot r_2 \cdot \omega(f, \delta_1, \delta_2),$$

where $\delta_1 = \max_{0 \le j \le n}(t_{j+1} - t_j)$ and $\delta_2 = \max_{0 \le k \le m}(t'_{j+1} - t'_j)$ and $\omega(f, \delta_1, \delta_2)$ is the bivariate modulus of continuity of the function f defined by

$$\omega(f, \delta_1, \delta_2) = \sup_{\substack{|x_1 - x_2| \le \delta_1 \\ |y_1 - y_2| \le \delta_2}} D(f(x_1, y_1), f(x_2, y_2)).$$

Proof. By (N1) and (N2), the results in [6] are easily extended to the bivariate case.

Better estimates can be obtained for fuzzy splines of Schoenberg type (for crisp Schoenberg splines see e.g. [15], [14]). Let the knots and auxiliary knots given as above, and $\xi_j = \frac{t_{j+1} + \ldots + t_{j+r_1-1}}{r_1 - 1}$, $j = -r_1 + 1, \ldots, n$, $\eta_k = \frac{t'_{k+1} + \ldots + t'_{k+r_2-1}}{r_2 - 1}$, $k = -r_2 + 1, \ldots, m.$ We define the bivariate fuzzy spline of Schoenberg type

$$S(f,x,y) = \sum_{j=-r_1+1}^{n} \sum_{k=-r_2+1}^{m} N_{j,r_1}(x) N_{k,r_2}(y) \cdot f(\xi_j, \eta_k).$$

Remark 4 If there are no basic knots in the interior of the interval $[0,1]$ then the bivariate fuzzy Schoenberg spline reduces to the bivariate fuzzy Bernstein polynomial similar to the crisp case, so the results of this paper extend the results in [8] and [10].

As in [14] we obtain the error bound in approximation by fuzzy splines of Schoenberg type:

Theorem 5 Concerning the error in approximation by fuzzy Schoenberg splines we have

$$D(S(f,x,y), f(x,y)) \le (1 + h(r_1, \delta_1))(1 + h(r_2, \delta_2))\omega(f, \delta_1, \delta_2),$$

where $h(r,\delta)$ is

$$h(r,\delta)=\min\left\{\frac{1}{\sqrt{2r-2}},\sqrt{\frac{r}{12}}\delta\right\}.$$

Proof. The proof is similar to the unidimensional case (see [6]).

This result can be seen as an error estimate for the digital terrain modelling method described in [2].

In what follows we discuss the usefulness of the method in digital terrain modelling. Uncertainty in digital terrain modelling is mainly due to measurement errors. These types of errors can be treated as probabilistic uncertainties, however, in the present case, repeating measurements in order to use statistical methods is maybe expensive and the model need not be very accurate.

Possibilistic (non-statistical) uncertainty may be introduced in the modeled system if we want to use our present model of the same terrain in the future. Even in near future, the model may change due to several reasons, such as erosion or some human factor. It is easy to remark that the effect of these changes may not be neglected in many cases and new measurements after relatively small changes are unnecessary and too expensive. Another possible treatment of this problem would be modelling of the future evolution of the system, which would imply a much more complex modelling problem, maybe too expensive from the computational point of view.

Conclusions

Approximation of bivariate fuzzy functions by bivariate fuzzy B-spline series have been studied and error estimates are obtained in terms of the modulus of continuity.

Since fuzzy B-spline series are already studied from a practical point of view for digital terrain modelling (see [2], [11]) their study from a theoretical point of view is motivated.

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