Absorbing Norms in the Family of Fuzzy Operators

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Abstract: Considering, that fuzzy applications are problem-oriented, there are many other operators used in problem solving beside of t-norms and t-conorms. In the paper a review of the relationships between uninorms, nullnorms, t-norms and conorms and absorbing norms is given. An example of absorbing norms is explained to stress that there are absorbing norms which are neither uninorms nor nullnorms.

Keywords: absorbing sorms, uninorms, nullnorms

1 Introduction

Since first fuzzy-based applications several classes of operations of intersection, union and complement operators, supported appropriate axioms, have been introduced. By accepting some basic conditions, a broad class of set of operations for union and intersection is formed by t-norms and t-conorms. For a kind of generalization of t-norms and t-conorms the concept of uninorm and nullnorms was introduced by Yager and Rybalov [1]. The structures of those operators on the unit square are similar, and they have representations based on t-norms and conorms, but .v ery important properties for applied operations in many applications are the absorbing, the evolutionary and the compensation ones. The conventional operators do not work well in lot of cases, but the absorbing norms satisfy that the neutral element e is increasing starting from zero till $e = 1$, and consequently the min operator is developing into the max operator. Beside this operators make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. It is important to stress that there are absorbing norms which are neither uninorms nor nullnorms.

A function $S:[0,1]^2 \to [0,1]$ is called *triangular conorm (t-conorm)* if and only if it fulfills the following properties for all $x,y,z \in [0,1]$:

(S1) $S(x,y)=S(y,x)$, i.e., the t-conorm is commutative,

(S2) $S(S(x,y),z)=S(x,S(y,z))$, i.e., the t-conorm is associative,

(S3) $x \le y \Rightarrow S(x,z) \le S(y,z)$, i.e., the t-conorm is monotone,

(S4) $S(x,0)=x$, i.e., a neutral element exists, which is 1.

A function $T:[0,1]^2 \to [0,1]$ is called *triangular norm (t-norm)* if and only if it fulfills the following properties for all $x,y,z \in [0,1]$

(T1) $T(x,y)=T(y,x)$, i.e., the t-norm is commutative,

(T2) $T(T(x,y),z)=T(x,T(y,z))$, i.e., the t-norm is associative,

(T3) $x \le y \Rightarrow T(x,z) \le T(y,z)$, i.e., the t-norm is monotone,

(T4) $T(x,1)=x$, i.e., a neutral element exists, which is 1.

When aggregating data in applications, it is convenient to assign to each tuple of elements in [0,1] a unique number, element in [0,1]. Generally this unique number is calculated by a function, which is a derivate from a binary operation on [0,1] (for instance, from t-norms and t-conorms).

An aggregation operator is a function $\mathbf{A}:\bigcup_{n\in N}[0,1]^n \to [0,1]$ such that:

(i) $\mathbf{A}(x_1,x_2,...,x_n) \le \mathbf{A}(y_1,y_2,...,y_n)$ whenever $x_i \le y_i$ for all $i \in \{1,2,...,n\}$

(*ii*) $\mathbf{A}(x)=x$ for all $x \in [0,1]$

(*iii*) $\mathbf{A}(0,0,...,0)=0$ and $\mathbf{A}(1,1...,1)=1$.

Each t-norm and each t-conorm is a commutative, associative aggregation operator with a neutral element 1 or 0, respectively. Moreover, a commutative, associative aggregation operator with a neutral element $e \in [0,1]$ is a t-norm if and only if $e=1$, and a t-conorm if and only if $e=0$, and a uninorm if and only if $e \in]0,1[$.

The most popular aggregation operators are the arithmetic mean, the geometric mean, the harmonic mean, quadratic mean, and the *p*-mean (see [4]).

$\mathbf{A}$ is an idempotent aggregation operator if and only if $T_M \leq \mathbf{A} \leq S_M$.

About aggregation operators on fuzzy sets see [3]. In [7] we find a review of *averaging* operators, and OWA operators used in fuzzy multicriteria decision making.

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (T1)-(T3) coincide for t-norms and (S1)-(S3) for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms.

Before the introduction of the exact meaning of the uninorms, some other operator groups were developed, which are related to uninorm properties:

Dombi aggregative operators (1982), which are representable uninorms,

Pseudo-additions, introduced by Pap (1995), where uninorms are particular cases, see Section 1.1. in [8].

Associative compensatory operators (see [9]), which are represetable uninorms.

A *uninorm* is a binary operation U on the unit interval, i.e., a function

$U:[0,1]^2 \to [0,1]$ which satisfies the following properties for all $x, y, z \in [0,1]$

(U1) $U(x,y) = U(y,x)$, i.e. the uninorm is commutative,

(U2) $U(U(x,y),z) = U(x,U(y,z))$, i.e. the uninorm is associative,

(U3) $x \leq y \Rightarrow U(x,z) \leq U(y,z)$, i.e. the uninorm monotone,

(U4) $U(e,x) = x$, i.e., a neutral element exists, which is $e \in [0,1]$.

The structure of uninorms was studied extensively in [4], [5], [6] where it was proved, among others, that there is no uninorm which is continuous on the whole unit square. It is evident that, for an arbitrary uninorm U with the neutral element $e \in]0,1[$, the operations $T_U, S_U : [0,1]^2 \to [0,1]$, which are defined by

$$T_U(x,y) = \frac{1}{e} U(ex,ey),$$

$$S_U(x,y) = \frac{1}{1-e}\left(U(e+(1-e)x, e+(1-e)y) - e\right),$$

are t-norms and t-conorms, respectively. This means that to any uninorm U with neutral element $e \in \left]0,1\right[$ corresponds a t-norm T_U and a t-conorm S_U such that

for all $(x, y) \in [0, e]^2$ we have $U(x, y) = T_U(x, y)$,

for all $(x, y) \in [e,1]^2$ we have $U(x, y) = S_U(x, y)$ and

in the rest of the unit square we have $\min((x, y) \le U(x, y) \le \max(x, y)$, i.e. U is bounded by the minimum and maximum.

Another variation of t-norms and t-conorms (again modifying only the axiom concerning the neutral element) has been recently proposed in [5] and [10].

A *nullnorm* is a binary operation V on the unit interval [0,1], i.e., a function $V : [0,1]^2 \to [0,1]$ which, for all $x, y, z \in [0,1]$, satisfies the following properties

(V1) $V(x, y) = V(y, x)$, i.e. the nullnorm is commutative,

(V2) $V(V(x, y), z) = V(x, V(y, z))$, i.e. the nullnorm is associative,

(V3) $x \le y \Rightarrow V(x, z) \le V(y, z)$, i.e. the nullnorm monotone,

(V4) there exist $a \in \left]0,1\right[$ such that $V(x,0) = x$ for all $x \in [0, a]$ and $V(x,1) = x$ for all $x \in [a,1]$.

It is immediately clear that each nullnorm V satisfies $V(x, a) = a$ for all $a \in [0,1]$, i.e., a is an annihilator of V. Similarly as for uninorms, for a nullnorm V with annihilator $a \in \left]0,1\right[$, the operations $T_V, S_V : [0,1]^2 \to [0,1]$ defined by

$$T_V(x, y) = \frac{1}{1-a}\left(V\left(a + (1 - ha)xae + (1 - a)y\right) - a\right),$$

$$S_V(x, y) = \frac{1}{a} V(ax, ay)$$

2 Absorbing Norms as the Special Operators beside Uninorms and Nullnorms

By accepting some basic conditions, a broad class of set of operations for union and intersection is formed by t-norms and t-conorms. Very important properties for applied operations in many applications are the absorbing, the evolutionary and the compensation ones, and conventional operators do not work well in lot of cases. The absorbing norms satisfy that the neutral element *e* is increasing starting from zero till *e* = 1, and consequently the min operator is developing into the max operator [2].

A function $A:[0,1]^2 \to [0,1]$ is called absorbing norm if and only if fulfils the following properties for all $x, y, z \in [0,1]$

(Aa1) There exists an absorbing element (annihilator) $a \in [0,1]$, i.e., $A(x,a) = a, \forall x \in [0,1]$.

(Aa2) $\mathrm{A(x,y)} = \mathrm{A(y,x)}$) that is, A is commutative,

(Aa2) A(A(x,y),z) = A(x,A(y,z)) that is, A is associative

T-norms and t-conorms are special absorbing-operators, namely for any t-norm *T*, $T(0,x) = 0, \forall x \in [0,1]$ and for any t-conorm *S*, $S(1,x) = 1, \forall x \in [0,1]$.

Like uninorms the structure of absorbing-norms is closely related to t-norms and t-conorms on the domains $[0,a] \times [0,a]$ and $[a,1] \times [a,1]$, namely:

The mapping $A_{\min}$: $[0,1]^2 \to [0,1]$ defined by

$$A_{\min}(x,y) = \begin{cases} \max(x,y), & \text{if } (x,y) \in [0,a] \times [0,a] \\ \min(x,y) & \text{elsewhere} \end{cases}$$

and the mapping $A_{\max}$: $[0,1]^2 \to [0,1]$ defined by

$$A_{\max}(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [a,1] \times [a,1] \\ \max(x,y), & \text{elsewhere} \end{cases}$$

are absorbing-norms with the absorbing element *a*.

The mapping $A_{\min}^{a} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_{\min}^{a}(x,y) = \begin{cases} a, & \text{if } (x,y) \in [0,a] \times [0,a] \\ \min(x,y), & \text{elsewhere} \end{cases}$$

and the mapping $A_{\max}^{a} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_{\max}^{a}(x,y) = \begin{cases} a, & \text{if } (x,y) \in [a,1] \times [a,1] \\ \max(x,y), & \text{elsewhere} \end{cases}$$

are absorbing-norms with absorbing element *a*.

The mapping $A_{a}^{\min} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_a^{\min}(x,y)=\begin{cases}\min(x,y), & \text{if } (x,y)\in[a,1]\times[a,1]\\ a, & \text{elsewhere}\end{cases}$$

and the mapping $A_a^{\max}:[0,1]\times[0,1]\to[0,1]$ defined as

$$A_a^{\max}(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[0,a]\times[0,a]\\ a, & \text{elsewhere}\end{cases}$$

are absorbing-norms with absorbing element *a*.

Assume that *A* is an absorbing-norm with absorbing element *a*. The dual operator of *A* defined as $\overline{A}(x,y)=1-A(1-x,1-y)$ is an absorbing-norm with absorbing element 1-*a*.

Let us define a kind of complements of $A_{\min}$ and $A_{\max}$ by replacing in the definitions the operator min with max and the max with min as follows:

$$\overline{(A_{\min})}^{\max}(x,y)=\begin{cases}\min(x,y), & \text{if } (x,y)\in[0,a]\times[0,a]\\ \max(x,y) & \text{elsewhere}\end{cases}$$

$$\overline{(A_{\max})}^{\max}(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[a,1]\times[a,1]\\ \min(x,y), & \text{elsewhere}\end{cases}.$$

We have received the first uninorms given by Yager and Rybalov

$U_c(x,y)=\overline{(A_{\max})}^{\max}(x,y)$, and $U_d(x,y)=\overline{(A_{\min})}^{\max}(x,y)$.

Following the construction given by Fodor, Yager and Rybalov for uninorms any t-norm *T* can be transformed to an absorbing-norm on $[a,1]\times[a,1]$ in the following manner.

Let *T* be any t-norm and define $T_A(x,y)=a+(1-a)T\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right),\ a\le x,y\le 1$.

It is easy to see that T_A has the properties of t-norms, and it is also an absorbing norm on [0, 1] with absorbing element *a*.

In a similar manner any t-conorm can be transformed to an absorbing-norm.

Let *S* be any t-conorm and define $S_A(x,y)=aS\left(\frac{x}{a},\frac{y}{a}\right)$ if $0\le x,y\le a$.

S_A has the properties of t-conorms, and it is also an absorbing norm on [0, 1] with absorbing element *a*.

Let S and T be a t-conorm and a t-norm, respectively. The mapping $A_{\min}^{ST}:[0,1]\times[0,1]\to[0,1]$

$$A_{\min}^{ST}(x,y)=\begin{cases} S_A(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ T_A(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \min(x,y), & \text{elsewhere} \end{cases}$$

and the mapping $A_{\max}^{ST}:[0,1]\times[0,1]\to[0,1]$

$$A_{\max}^{ST}(x,y)=\begin{cases} S_A(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ T_A(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases}$$

are absorbing-norms with absorbing element *a*.

Proposition in [4] proofs, that if A is a given nullnorm, then

$$S(x,y)=\frac{1}{a}A(ax,ay) \quad \text{if } 0\le x,y\le a\,,$$

$$T(x,y)=\frac{A(a+(1-a)x,\mathrm{a}+(1\text{-}\mathrm{a})y)-a}{1-a} \quad a\le x,y\le 1\,.$$

are t-norm and t-conorm, respectively.

Analogously to the results of Fodor et al. in [5] for uninorms it is possible to introduce the weakest and strangest absorbing-norms.

The mapping $A_W:[0,1]\times[0,1]\to[0,1]$

$$A_W(x,y)=\begin{cases} \max(x,y), & \text{if } (x,y)\in[0,a]\times[0,a] \\ 0, & \text{if } (x,y)\in\,]a,1]\times\,]a,1] \\ \min(x,y), & \text{elsewhere} \end{cases}$$

and the mapping $A_S:[0,1]\times[0,1]\to[0,1]$

$$A_S(x,y)=\begin{cases} 1, & \text{if } (x,y)\in[0,a[\,\times[0,a[\\ \min(x,y), & \text{if } (x,y)\in[a,1]\times[a,1] \\ \max(x,y), & \text{elsewhere} \end{cases}$$

are the weakest and strongest absorbing-norms, respectively, i.e. for any absorbing-norm A the following inequality holds:

$$A_W(x,y)\le A(x,y)\le A_S(x,y)\,.$$

It is important to stress that there are absorbing norms which are neither uninorms nor nullnorms.

Example

The mapping $A_a^{\min} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_a^{\min}(x, y) = \begin{cases} \min(x, y), & \text{if } (x, y) \in [a,1] \times [a,1] \\ a, & \text{elsewhere} \end{cases}$$

and the mapping $A_a^{\max} : [0,1] \times [0,1] \to [0,1]$ defined as

$$A_a^{\max}(x, y) = \begin{cases} \max(x, y), & \text{if } (x, y) \in [0, a] \times [0, a] \\ a, & \text{elsewhere} \end{cases}$$

are absorbing norms, but are not uninorms and are not nullnorms.

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