Conditionally distributive real semirings

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Abstract: A characterization of all pairs (U,S) where is U left-continuous uninorms with neutral element e from (0,1] and S countinuous t-conorms satisfying distributivity of U over S (so-called conditional distributivity) is given.

Keywords: t-norm, t-conorm, uninorm, distributivity, semiring.

1. Introduction

Many different kinds of operation defined on subset of real numbers play fundamental roles in many important fields as for exmple in fuzzy set theory, fazzy logic, neural nets, operation research, optimization problems, differential equations etc. Special intention is paid to operations defined on interval of reals. The examples are t-norms and t-conorms which act on the interval [0,1], pseudo-additions and pseudo-multiplications in the sense of Sugeno and Murofushi [9] which act on the interval $[0,\infty]$ or in the sense of E. Pap [6] which act on the interval [a,b] where is [a,b] is closed subinterval of $[-\infty, +\infty]$, compesatory operators (Klement, Mesiar, Pap [5]) and uninorms (Fodor, Yager, Rybalov [1]).

In this paper we will consider a binary operations on unit interval i.e. above mentioned t-norms, t-conorms and uninorms.In many situations those operations have to be distributive one with respect to other as for example in the context of integrals based on decomposable measures. If the range of pseudo-aditive measures is subset of the unit interval [0,1], countinous t-conorms S are natural candidates for the pseudo-adition, leading to the concept of S-decomposable measures. For generalized Lebesque integral for [0,1]-valued functions, a second operation U turning ([0,1], S,U) into a semiring will be considered. Consequently U should be comutative, associative, non-decreasing, and should have a neutral element e from (0,1] i.e. it should be a uninorm (

if $e \in (0,1)$) or a t-norm (if e =1). Also some distributivity of U over S (so-called conditional distributivity) will be required (so [0,1], S, U) becomes a conditionally distributive semiring.

2. Preliminaries

Triangular norms and conorms were originally introduced in the context of probablistic metric spaces. A triangular norm (shortly t-norm) i a binary operation on the unit interval which is commutative, associative, non-decreasing in each component and which has 1 as a neutral element. Dually, a triangular conorm (shortly t-conorm) is a binariy operation on the unit interval which is commutative ,associative, non-decreasing in each component, and which has 0 as a neutral element. . The most important t-norms are the minimum T_M , the product T_P, and the Lukasiewicz t-norm T_L, which (together with the corresponding t-conorms maximum S_M, probablistic sum S_P, and Lukasiewicz t-conorm S_L) are given by

$$T_M(x,y) = \min(x,y), \qquad S_M(x,y)=\max(x,y)$$
$$T_P(x,y) = xy \qquad S_P(x,y)= x+y-xy$$
$$T_L(x,y) = \max(x+y-1,0) \qquad S_L(x,y) = \min(x+y,1)$$

Each countinuous Archimedian t-norm T has a multiplicative generator i.e. a countinnuous, strictly increasing function $\Theta:[0,1]\to[0,1]$ satisfying $\theta(1)=1$ such that $T(x,y)=\Theta^{(-1)}(\Theta(x)\Theta(y))$ where $\Theta^{(-1)} : [0,1]\to[0,1]$ is the pseudo-inverse of Θ given by $\Theta^{(-1)}(x) = \Theta^{-1}(\min(x,\Theta(1))$

The countinuous, strictly increasing functions $s : [0,1]\to[0,\infty]$ satisfying $s(0)=0$ serve as aditive generators of countinuous Archimedean t-conorms S as follows: $S(x,y)=s^{(-1)}(s(x)+s(y))$. In particular , S is strict if and only if $s(1)=\infty$ (i.e. if s is bijection) and S is nilpotent if and only if $s(1)<\infty$.

Each continuous t-norm(t-conorm) can be represented as an ordinal sum of continuous Archimedean t-norms(t-conorms) i.e. there exists a uniquely determined (finite or coutable infinite) index set A, a family of uniquely determined pairwise disjoint open subintervals $(a_\alpha, b_\alpha)_{\alpha\in A}$ of [0,1] and a family of uniquely determined continuous Archimedean t-norms(t-conorms) $(T_\alpha)_{\alpha\in A}$ such that $T=(<a_\alpha,e_\alpha,T_\alpha>)_{\alpha\in A}$ where each $<a_\alpha,e_\alpha,T_\alpha>$ is called summand.

A third class of operations will be important for us the so-called uninorms. Uninorrms are generalizations of t-norms and t-conorms allowing the neutral element lying anywhere in the unit interval [0,1]. Therefore a uninorm is binary operation on the unit interval which is commutative, associative, non-decreasing in each component and which has a neutral element e from [0,1]. Suppose U is a uninorm with neutral element e from (0,1). Define two functions T_U and S_U on the unit square as follows

$$T_U(x,y)=\frac{U(ex,ey)}{e} \quad x,y\in[0,1] \tag{1}$$

$$S_U(x,y)=\frac{U(e+1-e)x,e+(1-e)y)-e}{1-e} \quad x,y\in[0,1] \tag{2}$$

It is easy to verifay that T_U defined by (1) is a t-norm and S_U defined by (2) is a t-conorm. Therefore the structure of uninorms on the squares $[0,e]^2$ and $[e,1]^2$ is closely related to t-norms and t-conorms. That is we have

$$U(x,y)=eT\left(\frac{x}{e},\frac{y}{e}\right) \quad if\ 0\le x,y\le e \tag{3}$$

and

$$U(x,y)=e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) \quad if\ e\le x,y\le 1 \tag{4}$$

with some t-norm T i some t-conorm S. T is called the underlying t-norm of U and S is called the underlying t-conorm of U.

Concerning the definicion of U on the rest of unit square we have that

$$\min(x,y)\le U(x,y)\le \max(x,y) \text{ if } x\le e\le y \text{ or } x\ge e\ge y$$

Each increasing bijection $f:[0,1]\to[0,\infty]$ defines (using the convention $0\cdot\infty=0$) a left-countinuous uninorm U (there exist no continuous uninorm) with neutral element $f^{-1}(1)$ by means of $U(x,y)=f^{-1}(f(x)f(y))$.

The generators of t-conorms, t-norms and uninorms suggest that a t-conorm can be seen as transformations of the addition of non-negative real numbers, wheres uninorms and t-norms are transformations of multiplications.

Throughout this paper we shall work with a continuous t-conorm S and left-contninuous uninorm U satisfying the following conditional distributivity (CD)

U(x,S(y,z))=s(U(x,y),U(x,z)) for all x,y,z from [0,1] with S(y,z)<1 (CD)

In this context we shall refer to ([0,1],S,U) as a conditionally distributive semiring

3. Conditional distributivity

In this section we will consider two cases depending on neutral element e of uninorm U. The first case is when e=1 and then U becomes a t-torm T. The second case is when $e\in(0,1)$.

3.1 Conditional distributivity of t-norm T over t-conorm S

The first case is described in the following theorem whose proof can be found in [3]

Theorem 1 *A continuous t-norm T and countinuous t-conorm S satisfies the condition (CD) if and only if we have either one of the following cases*
(i) $S=S_M$
(ii) There is a strict t-norm T^* *and a nilpontent t-conorm* S^* *such that the additive generator s of* S^* *satisfaying s(1)=1 is also a multiplicative generator of* T^*, *and there is an a* $\in [0,1[$ *such that for some countinuous t-norm* T^{**}, *we have* $T=(<o,a,T^{**}>,<a,1,T^*>)$ *and* $S=(<a,1,S^*>)$.

Remark 1 If in the functional equation (CD) we omit the condition S(y,z)<1 we say that T is distributive over S and then we have only trivial solutions, i.e., $S=S_M$.

This remark shows how much the distributivity lows restrict the choice of possible t-conorms. Thus it seems reasonable to restrict the domain of the distributivity low if we look for solutions which are not trivial.

A full characterization of all pairs (T,S) satisfying the condition (CD) which are not continuous is still an open problem.

3.2 Conditional distributivity of uninorm U over t-conorm S

In this subsection we give a characterization of all pairs (U,S) satisfying (CD) where U is a left-continuous uninorm with neutral element $e \in (0,1)$ and S is a continuous t-conorm. In this subsection we will distinguish two cases. The first is when neutral element e of the uninorm U is an idempotent element of the t-conorm S. The second case is when neutral element e of the uninorm U is not an idempotent element of the t-conorm S.

Theorem 2 *A left-continuous uninorm U with neutral element* $e \in (0,1)$ *and a continuous t-conorm S where e is an idempotent element of S satisfy (CD) if and only if* $S=S_{M..}$

Proof Obviously each uninorm U is distributive over S_M because of monotonicity of U. Conversely if U is conditionally distributive over S when neutral element e of the uninorm U is an idempotent element of S we have for all $x \in [0,1]$, since e<1 and S(e,e)=e, the following:

$$x=U(x,e)=U(x,S(e,e))=S(U(x,e),U(x,e))=S(x,x).$$

Therefore each element from [0,1] is an idempotent of S, and so S must be a max operator.▮

The second case is more complicated and in order to investigate it we shall prove a sequence of lemmas. Firstly we present a lemma in which the ordinal sum structure for a continuous t-conorm simplifies considerablly.

Lemma 1 *Let U be a left-continuous uninorm with neutral element $e \in (0,1)$ and let S be a continuous t-conorm for which e is not an idempotent element. If the pair (U,S) satisfies the condition (CD), then $|A|=1$.*

Proof : Since the neutral element e is not an idempotent element of S then $e \in (a_\alpha, e_\alpha)$ for some $\alpha \in A$. We will prove that each $x \geq e_\alpha$ and each $x \leq a_\alpha$ is idempotent of S. We know that $S(a_\alpha, a_\alpha) = a_\alpha$ and $S(e_\alpha, e_\alpha) = e_\alpha$ hold. Assume that $e_\alpha<1$. The case when $e_\alpha=1$ is trivial. Since (CD) holds we have for all $x \in [0,1]$ the following:

$$U(x,e_\alpha)=U(x,S(e_\alpha,e_\alpha))=S(U(x,e_\alpha),U(x,e_\alpha)).$$

Therefore for each x in [0,1] $U(x,e_\alpha)$ is an idempotent element for S. Now observe the continuous function $U(x,e_\alpha)$ for $x \geq e$. It is a continuous increasing function from [e,1] into $[e_\alpha,1]$. Thus each element from $[e_\alpha,1]$ is an idempotent element of S.
Using similar arguments we will show that each element $x \leq a_\alpha$ is an idempotent of S. Assume that $a_\alpha>0$. The case when $a_\alpha=0$ is trivial. We have because of (CD) for all x from [0,1] the following:

$$U(x,a_\alpha)= U(x,S(a_\alpha,a_\alpha))=S(U(x,a_\alpha),U(x,a_\alpha))$$

Therefore for all x from [0,1] $U(x,a_\alpha)$ is an idempotent element of S. Now observe the continuous function $U(x,a_\alpha)$ for $x \leq e$. It is a continuous increasing function from [0,e] into $[0,a_\alpha]$. Thus each element in $[0,a_\alpha]$ is an idempotent of S. From the previous considerations we can conclude that $|A|=1$, i.e., $S=(\langle a,b,S^*\rangle)$ where S^* is a continuous Archimedean t-conorm.▌

Lemma 2 *Let U be a left-continuous uninorm with neutral element $e \in (0,1)$ and S be a continuous t-conorm for which e is not an idempotent element . If the pair (U,S) satisfies the condition (CD), then $U(x,y) \in [a,b]$ for all $x,y \in [a,b]$, where a, b are from the previous Lemma 1 such that $S=(\langle a,b,S^*\rangle)$.*

Proof: We will prove that a,b are idempotent elements for U. From the representation of continuous t-conorm we know that S* is an Archimedean continuous t-conorm which is either strict or nilpotent. Firstly we consider the case when S* is strict.

(i) If S* is strict then we have that $S(e,e)<1$ because $e<1$. Let us take the points b and e and apply (CD) whence obtain the following:

$$U(b,S(e,e))=S(U(b,e),U(b,e))=U(b,e)=b$$

By induction we get for all n from N

$$b= U(b,e)=U(b, e_S^{(2^n)})$$

Since $\lim_{n\to\infty} e_S^{(2^n)} = b$, this together with continuity of U on $[e,1]^2$, implies b=U(b,b)

Let us prove now that U(a,a)=a. Similary as in the previous case we will take the points a and e and again apply the condition (CD) whence obtain the following:

U(a,S(e,e))=S(U(a,e),U(a,e))=U(a,e)=a and consequently we get U(a, $e_S^{(\frac{1}{2})}$) =U(a,e)=a (see remark 3.5 in [3]). By induction we get for all n from N

$$U(a, e_S^{(2^{-n})}) = U(a,e)=a.$$

Since $\lim_{n\to\infty} e_S^{(2^{-n})}) = a$ this together with continuity of S and U on $[0,e]^2$ implies U(a,a)=U(a,e)=a

(ii) Let now S* be a nilpotent t-conorm and fix c=inf{x $\in [0,1] \mid S(x,x)=1$ }<1. Then immeadieately we get S(b,b)=1=b. Therefore it remains only to show U(a.a)=a.

If e<c then S(e,e)<1 and in the same manner as in the strict case we get U(a,a)=a.

If c≤e then for x<c holds S(x,x)<1 and applying (CD) on the points a and x we have the following:

$$U(a,S(x,x))=S(U(a,x),U(a,x))=U(a,x).$$

When x $\to c$ because of continuity of S and U on $[0,e]^2$ we have

U(a,S(c,c))=U(a,1)=U(a,c). By induction we get

U(a, $c_S^{(2^{-n})}) = U(a,c)$ for all n from N implying U(a,a)=U(a,1) $\geq a$. Opposite inequality is trivial because U(a,a) $\leq U(a,e) = a$,Therefore U(a,a)=a in this case too.▌

So far we have seen that when (CD) is satisfied then ordinal sum representation for t-conorm S is simplfied because we have only one summand <a,b,S*>. Also we have showed that U(x,y) $\in [a,b]$ when x,y $\in [a,b]$, i.e., uninorm U is compatibile with structure of t-conorm S.

Now we can apply results from [4]

Theorem 3 *Let ($[0,1], U, S$) be a conditionally distributive semiring*

(i) If S is strict t-conorm, i.e if it is generated by a bijective additive generator s:[0,1]→[0,∞], then U is generated by c·s for some constant c from (0,∞) and hence has the neutral element $s^{-1}(\frac{1}{c})$

(ii) If S is a nilpotent t-conorm, i.e., if it has a (unique) additive generator s which can be seen as an increasingbijection s:[0,1]→[0,1], then U is a t-norm with multiplicative generator s.

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