Real relation
New approach to Fuzzy relation

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Abstract: Fuzzy relation is fuzzy subset of finite Cartesian power of a given set, i.e. a fuzzy set of n-tuples of n-elements of analyzed set. It is known that all laws of classical sets algebra is not satisfied in the fuzzy set theory. Real relation (R-relation) is based on real set (R-set) theory. R-set theory is a consistent generalization of classical set theory, contrary to fuzzy set theory. In R-set theory range of membership function is real interval [0, 1] as in the fuzzy set theory. The basic difference between R-set theory and fuzzy set theory lies in the fact that all laws of classical set algebra are valid in the R-set theory. As a consequence all fundamental properties of classical relations are preserved in the generalized case - real relations.

Keywords: Fuzzy sets, R-sets, Relations, R-relations

1 Introduction

Formally, a relation is a set - subset of finite Cartesian power of a given set. Fuzzy relation (F-relation) is fuzzy set [1]. All known theories of fuzzy sets are realization of *multi-valued (MV) logics* [2]. In MV logics are not preserved all fundamental properties (logical laws) of classical logic. Consequence is that all laws of set algebra are not preserved in fuzzy set theory.

In this paper is introduced real relation (R-relation). R-relation is based on real set (R-set) theory. R-set theory is a consistent generalization of classical set theory, contrary to theories of fuzzy sets. The range of membership function in R-set theory as in fuzzy set theory is a real interval [0, 1]. The basic difference between R-set theory and fuzzy set theory lies in the treatment of predicates (properties, relations etc.). In classical logic and/or classical set theory value (of truth, or of memberships) is equivalent with predicate itself, and in calculus values are enough. In classical case this presumption is correct since Boolean algebra of predicates are homomorphically mapped in reduced two values Boolean algebra. Fuzzy approach use same presumption as classical approach.. Problem is in the fact that by mapping Boolean logic of predicates in set with more them two elements cant be Boolean logic. So, treating the values of membership functions of same predicate as predicate itself the nature of predicate can be preserved in general case.

In the case of R-sets there are two levels: (a) symbolic (logical) deal with predicates as a quality independent of value realization and this level is structured by Boolean algebra; and (b) valued level (non-logical) deal with convex interpolations.. Since logical nature of R-set theory is a meter of symbolic level with Boolean algebra, all laws of classical set algebra are valid in the R-set theory and all fundamental properties of classical relations are preserved in the R-relations. Since R-relations are generalization of relations, they have additional properties and capabilities, too.

In Section 2 basic notion of relation and fuzzy relation is given. Real relations are explained and illustrated in Section 3.

2 Relation and fuzzy relation

2.1 Relation

Relation is a subset of a finite Cartesian power $A^n = A \times \cdots \times A$ of a given set A, i.e. a set of n-tuples $(a_1, ..., a_n)$ of n elements of A.

A subset is called an n-place, or an n-ary, relation on A. The number n is called the rank, or type, of the relation R. A subset $R \subseteq A^n$ is also called an n-place, or n-ary, predicate on A. The notation $R(a_1, ..., a_n)$ signifies that $(a_1, ..., a_n) \in R$.

One-place relations are called properties. Two-place relations are called binary, three-place F-relations are called ternary, etc.

The set A^n and the empty subset $\emptyset$ in A^n are called, respectively, the universal relation and the zero relation of rank n on A^n.

If R and P are n-place relations on A, then the following subsets of A^n will also be n-place relations on A:

$$R \cup P, \qquad R \cap P, \qquad R' = A^n \setminus R, \qquad R \setminus P.$$

The set of all n-ary relations on A is a Boolean algebra relative to the operations $\cup, \cap, '$.

2.2 Fuzzy relation

Fuzzy relation (F-relation) is generalization of relation so that the rang of membership function is interval $[0, 1]$. F-relation is a fuzzy subset of a finite Cartesian power $A^n = A \times \cdots \times A$ of a given set A, i.e. a fuzzy set of n-tuples $(a_1, ..., a_n)$ of n elements of A.

If R and P are n-place F-relations on A, then the following F-subsets of A^n will also be n-place relations on A:

$$R \cup P, \qquad R \cap P, \qquad R' = A^n \setminus R, \qquad R \setminus P.$$

where:

$$\begin{aligned}
(R \cup P)(a_1, ..., a_n) &= T(R(a_1, ..., a_n), P(a_1, ..., a_n)), \\
(R \cup P)(a_1, ..., a_n) &= S(R(a_1, ..., a_n), P(a_1, ..., a_n)), \\
R'(a_1, ..., a_n) &= N(R(a_1, ..., a_n)), \\
(R \setminus P)(a_1, ..., a_n) &= \begin{cases} R(a_1, ..., a_n) - P(a_1, ..., a_n), & R - P > 0 \\ 0, \qquad R(a_1, ..., a_n) - P(a_1, ..., a_n) \leq 0 \end{cases}, \\
(a_1, ..., a_n) &\in R, P \subseteq A^n,
\end{aligned}$$

T and S are T-norm and S-norm (T-conorm) respectively and N generalized negation.

The set of all n-ary F-relations on A is not a Boolean algebra relative to the operations $\cup, \cap, '$. As a consequence, laws of classical relation algebra are not valid in fuzzy case. So, in general case, the results based on classical relations couldn't be generiliezed and/or fuzzifed directly.

3 Real relation

Real relation (R-relation) is consistent generalization of relation, contrary to fuzzy relations. All laws of classical relation algebra are preserved in the case of R-relations, of course with richer interpretations. The algebra of classical relation and/or set-theoretic operations are based on truth functional principle from classical logic. It means that the value characteristic function of combined relation can be directly calculated by the values of characteristic function of relation components

The R-relation theory is an interpretation of Syntactic Structured and Semantic Convex (S^3C) logic [3]. S^3C logic is a consistent generalization of Boolean logic, contrary to known MV logics.

The fundamental notions of the theory of R-relations are: (a) the universe of discourses (set of objects, which can be in relations) and (b) relations (interaction between objects of universe). R-relation (or a predicate) creates a R-set in universe. A R-relation is primary or combined. A combined relation is a result of set operations.

Theory of R-relation has two levels: (a) *Symbolic*: computing with quality (nature) of relations and (b) *Valued*: computing with intensity (quantity) of relations (occupied with the intensities of relation). Qualitatively aspects of relations are meter of symbolic level, and algebra of relations on symbolic level is Boolean algebra. Quantitatively aspect of relation intensities - value of interaction between elements in analyzed tuple of relation is a meter of valued level and mathematical tool is convex interpolation.

3.1 Symbolic level: Boolean algebra

Characterization of a R-relation is a meter of symbolic (syntactic) level. A characteristic is a set function - a logical structure of R-relation. The domain of logical structure of a R-relation is a set of primary R-relations and all their subsets including an empty set (power set of primary R-relations), and range is a $\{0,1\}$-valued set. Any R-relation (primary and/or combined) is uniquely characterized by its logical structure. Logical structure is isomorphic mapping of R-relations set into set of R-relation structures. Since algebra of R-relations (on symbolic level) is Boolean algebra follows that algebra of R-relation structures set is also Boolean algebra.

3.1.1 R-relation

• *Constant R-relations* are classical (a) *universal relation* **1** (or A^n) and (b) *zero relation* **0** (or $\emptyset$); produce A^n and $\emptyset$ on the value (semantic) level, respectively.

• *A primary n-ary R-relation* R produced a primary R-subset on a valued level, $R \subseteq A^n$. The set of all primary R-relations is *the alphabet of primary R-relations.* We analyze only a finite alphabet $\Omega = \{R_1, ..., R_n\}$.

• *A combined R-relation* can be formed by using set operators ($\cap$ intersection, $\cup$ union and ^{c} complement) and rules.

• Rules for forming well formed R-relations (wfr) - are:

(a) Primary and constant R-relations are wfr ;

(b) If R and B are wfr-s then $(R \cap B)$, $(R \cup B)$, R^c are wfr-s ;

(c) There are no other wfr.

• *Basic (atomic) R-relations* $\mathbf{bR}(S)$, $S \in \mathcal{P}(\Omega)$ are defined by the following expressions:

$$\begin{aligned} \mathbf{bR}(S) &= \bigcap_{R_i \in S} R_i \bigcap_{R_j \in \Omega \setminus S} R_j^c; \\ S &\in \mathcal{P}(\Omega). \end{aligned}$$

• Any R-relation $R(R_1, ..., R_n)$ (set function) can be expressed as a function of basic R-relations in canonical union (disjunctive) form

$$R(R_1, ..., R_n) = \bigcup_{S \in \mathcal{P}(\Omega)} R\left(R_1^S, ..., R_n^S\right) \mathbf{bR}(S)$$

where:

$$R_i^S = \begin{cases} 1 & R_i \in S \\ 0 & R_i \notin S \end{cases} ; \quad S \in \mathcal{P}(\Omega), \quad i = 1, ..., n.$$

3.1.2 Structure of R-relation

The unique characteristic of wfr is their structure. *A structure of wfr* **is a set function:**

$$s : \mathcal{P}(\Omega) \rightarrow \{0,1\}.$$

where: $\mathcal{P}(\Omega)$ is a power set (a set of all subsets including an empty set) of alphabet Ω - set of primary R-relations.

3.1.3 Structure of R-relation constants:

Definition 1: *Structure functions of constant R-relations are the following set functions:*

$$s(\mathbf{1})(S) = 1 \qquad S \in \mathcal{P}(\Omega),$$

and

$$s(\mathbf{0})(S) = 0 \qquad S \in \mathcal{P}(\Omega).$$

3.1.4 Structure of primary R-relation

Structure of primary R-relation is defined in the following way:

Definition 2: *Structure function s of primary R-relation $R \in \Omega$ is the following set function [4]:*

$$s(R)(S) = \begin{cases} 1 & R \in S \\ 0 & R \notin S \end{cases} ; \qquad S \in \mathcal{P}(\Omega);$$

where: $\mathcal{P}(\Omega)$ *is a partitive set of* Ω - set of primary R-relations.

3.1.5 Structure of basic R-relation

The structure function of basic R-relation is *a basic structure function (structure for short)*. From the definition of basic R-relation $\mathbf{bR}(S)$, $S \in \mathcal{P}(\Omega)$; only one component of basic structure is equal to 1 (all others are 0):

$$s(\mathbf{bR}(S_1))(S_2) = \begin{cases} 1 & S_1 = S_2 \\ 0 & S_1 \neq S_2 \end{cases} ; \quad S_1, S_2 \in \mathcal{P}(\Omega).$$

In the case $|\Omega| = n$, the number of basic R-relations is 2^n.

Very important characteristics of basic R-relation structures $s(\mathbf{bR}(S))$, $S \in \mathcal{P}(\Omega)$; are:

(a) distinctness:

$$\begin{aligned} s(\mathbf{bR}(S_1) \cap \mathbf{bR}(S_2))(S) &= s(\mathbf{bR}(S_1))(S) \wedge s(\mathbf{bR}(S_2))(S) \\ &= 0, \quad S_1 \neq S_2, \\ S_1, S_2, S &\in \mathcal{P}(\Omega); \end{aligned}$$

(b) compactness:

$$\bigcup_{S \in \mathcal{P}(\Omega)} s(\mathbf{bR}(S))(S_i) = 1, \quad \forall S_i \in \mathcal{P}(\Omega).$$

3.1.6 Principle of structural functionality

The fundamental principle of S^3C logic and as a consequence of the new approach is *the principle of structural functionality.* The principle of structural functionality *[4]* says that the structures (values of structure components) of compounded R-relations uniquely determine the structure (values of structure components) of compound R-relation. This is achieved by defining *the structure function of connectives* as follows:

	$\neg$
1	0
0	1

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

where: $\wedge$ is the structure function of $\cap$; $\vee$ of $\cup$,and $\neg$ of ^{c}.

Domain of structure function of connectives is range of structural functions (not truth values!)

Using this, each structure function s extends uniquely to a structure determination of all R-relations as follows:

$$s\left(R^c\right)(S) = \neg s\left(R\right)(S),$$

$$s\left(R_1 \cap R_2\right)(S) = \left(s\left(R_1\right)(S) \wedge s\left(R_2\right)(S)\right),$$

$$s\left(R_1 \cup R_2\right)(S) = \left(s\left(R_1\right)(S) \vee s\left(R_2\right)(S)\right),$$

where: $S \in \mathcal{P}(\Omega)$.

Set of all R-relations, on symbolic level, generated by primary R-relations Ω operators $\cup, \cap, ^c$ is Boolean algebra. Structure function isomorphically maps R-relations into their structures. So, the set of structures of R-relations with $\vee, \wedge, \neg$ is Boolean algebra, defined on symbolic level.

3.1.7 Structure of combined R-relation

Any R-relation $R(R_1, ..., R_n)$ can be described as a set function of basic R-relations and its structure functions as follows:

$$R(R_1, ..., R_n) = \bigcup_{S \in \mathcal{P}(\Omega)} s(R)(S)\,\mathbf{bR}(S).$$

The number of possible R-relations in the case $|\Omega| = n$ is 2^{2^n}.

After R-relations are structured on symbolic level the next step is quantitative characterization of R-relations on valued level.

3.2 Valued level: convex interpolation

A fundamental notion on a valued level is *the universe of discourses* A, the universe for short. The number of elements in the universe is finite. R-relation R is real subset $R \subseteq A^n$ of finite Cartesian power of the universe A. R-relation is determined from valued point of view by intensities or values of R-relation membership function, defined on Cartesian power of the universe of discourse.

3.2.1 Intensity of R-relation

Every element from the Cartesian power of the universe A^n has the intensity of analyzed R-relation. Values of R-relation intensity in classical set theory is from valued set $\{0, 1\}$, but in the most general case is from real set $[0, 1]$. Any R-relation, from syntactic level, define corresponding set (classic and/or real) on the power of the universe.

Intensities of universal R-relation $\mathbf{1}$ (or A^n) and zero R-relation $\mathbf{0}$ (or $\emptyset$) are:

$$\begin{aligned} \mathbf{1}(a_1, ..., a_n) &\equiv 1, \quad \forall a_i \in A; \\ \mathbf{0}(a_1, ..., a_n) &\equiv 0, \quad \forall a_i \in A, \end{aligned}$$

respectively.

Primary R-relation $R \subseteq A^n$ and $R \in \Omega$ has components with intensities from real interval $[0, 1]$:

$$R : A^n \to [0, 1].$$

The nature of primary R-relation define intensity of component - value of interaction of analyzed n-tuple.

For determining the intensities of elements (values of interaction of order elements in n-tuples) of combined R-relations we have to introduce basic (atomic) functions.

3.2.2 Generalized product

A basic function defines the intensities (values of interaction of order elements in n-tuples) of basic R-relation. For purpose of definition of basic function we first define the generalized product $\otimes$.

Definition 4: *A generalized* $n-$*product is a binary operation* $\bigotimes_{(n)} : [0, 1]^2 \to [0, 1]$, *such that for all* $a_1, ..., a_n \in [0, 1]$ *the following five axioms hold: 1.commutativity; 2. associativity; 3. monotonicity ; 4. boundary condition (definition of t-norm [5] and new 5. non-negativity condition:*

$$\sum_{A \in \mathcal{P}(\Omega \setminus S)} (-1)^{|A|} \bigotimes_{a_i \in S \cup A} a_i \geq 0 \qquad \forall S \in \mathcal{P}(\Omega).$$

where: $\Omega = \{a_1, ..., a_n\}$.

3.2.3 Basic function

Basic R-functions $\mathbf{bR}(S) : A^n \to [0, 1]$, $S \in \mathcal{P}(\Omega)$ are defined in the following way:

Definition 5: *A basic (atomic) R-function is:*

$$\mathbf{bR}(S)(a_1, ..., a_n) = \sum_{R \in \mathcal{P}(\Omega \setminus S)} (-1)^{|R|} \bigotimes_{R_i \in S \cup R} R_i(a_1, ..., a_n)$$

where: $S \in \mathcal{P}(\Omega)$, $a_1, ..., a_n \in A$ *and* $\bigotimes$ *is the operator of generalized product.*

The characteristics of basic functions are:

(a) The sum of values of basic functions is identical to 1:

$$\sum_{S \in \mathcal{P}(\Omega)} \mathbf{bR}(S)(a_1, ..., a_n) = 1; \quad \forall a_1, ..., a_n \in A.$$

(b) The intensity of intersection of two different basic R-relations is identical to 0:

$$(\mathbf{bR}(S_i) \cap \mathbf{bR}(S_j))(a_1, ..., a_n) = 0; \quad \forall a_1, ..., a_n \in A,$$

where: $S_i \neq S_j$ and $S_i, S_j \in \mathcal{P}(\Omega)$.

3.2.4 Intensity of combined R-relation

A combined R-relation in a general case can be described by the structure of R-relation and basic functions:

$$R(R_1, ..., R_n) = \bigcup_{S \in \mathcal{P}(\Omega)} s(R)(S)\,\mathbf{bR}(S).$$

Intensity of combined R-relation, $R(R_1, ..., R_n) : A^n \to [0,1]$, in tuple $(a_1, ..., a_n)$ is the superposition of intensities for the same tuple of relevant basic R-relations, determined by its structural functions:

$$R(R_1, ..., R_n)(a_1, ..., a_n) = \sum_{S \in \mathcal{P}(\Omega)} s(R)(S)\,\mathbf{bR}(S)(a_1, ..., a_n),$$

where: $a_1, ..., a_n \in A$.

4 Example: Constructing real preference structures

(As a consequence of consistencies of S^{3}C fuzzy logic in the Boolean means, the generalized theory of sets based on this logic has all basic properties of set operations as classical set theory.

The application of S^{3}C logic for fuzzy set operations is illustrated on the fuzzy preference structures. Contrary to other approaches to the fuzzy preference structures [7], fuzzification (generalization) of the classical preference structure, based on the new logic and/or new fuzzy sets theory is straightaway.)

Preference structures are very important notions in the aria of Decision Making (DM).

4.1 Preference structures as a classical relations

Consider a set of alternatives A and suppose that a decision maker wants to judge them by pairwise comparison using fuzzy binary relations (P, I, J), where: *the strict preference relation* P, the *indifference relation* I and the *incomparability relation* J. For any $(a, b) \in A^2$, we classify:

$(a, b) \in P \Leftrightarrow$ DM prefers a to b;
$(a, b) \in I \Leftrightarrow$ a to b are indifferent to DM;
$(a, b) \in J \Leftrightarrow$ DM is unable to compare a and b.

The triplet (P, I, J) defined above satisfies the conditions formulated in the following definition of preference structure.

Definition 6: *[6] A preference structure on A is a triplet (P, I, J) of binary relations in A that satisfy:*
(P1) P is irreflexive, I is reflexive and J is irreflexive;
(P2) P is asymmetrical, I is symmetrical and J is symmetrical;
(P3) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$;
(P4) $P \cup P^t \cup I \cup J = A^2$.

Condition (P4) is called the completeness condition and this condition can be written equivalently in at least eight different ways:

(1) $co\,(P \cup I) = P^t \cup J$;
(2) $co\,(P \cup J) = P^t \cup I$;
(3) $co\,(P \cup P^t) = I \cup J$;
(4) $coP^t \cap coI \cap coJ = P$;
(5) $coP \cap coI \cap coJ = P^t$;
(6) $coP \cap coP^t \cap coJ = I$;
(7) $coP \cap coP^t \cap coI = J$;
(8) $P \cup P^t \cup I \cup J = A^2$.

It is possible to associate a single reflexive relation to any preference structure so that it completely characterizes this structure. The binary relation

$$R = P \cup I$$

is called the large preference relation of a given preference structure (P, I, J); R is always reflexive. Conversely, given a reflexive binary relation R in A, we can construct a preference structure on A as follows.

Proposition 1: *[6] Consider a reflexive binary relation R in A. The triplet (P, I, J) of binary relations in A constructed as follows:*
(i) $P = R \cap coR^t$;
(ii) $I = R \cap R^t$;
(iii) $J = coR \cap coR^t$,
is a preference structure on A such that $R = P \cup I$.

Any preference structure can be reconstructed from its large preference relation in the above way. For this reason, the large preference relation is sometimes called the characteristic relation.

Proposition 2: *[6] Consider a preference structure (P, I, J) on A and its large preference relation R. Then it holds that:*

$$(P, I, J) = \left(R \cap coR^t, R \cap R^t, coR \cap coR^t\right).$$

It is possible to associate a single reflexive relation R to any R-preference structure (P, I, J) so that it completely characterizes this structure.

Because fuzzy relations allow expressing degrees of preference, indifference or incomparability, it is very natural that fuzzy relation have been heavily involved in preference models [7]. Study of fuzzy preference structures has long tradition ([7]). Having at hand the classical concept of preference structures [8], one tries to fuzzify it in such a way that the most important classical characterizations fined a fuzzy counterpart. The study of fuzzy preference structures, is an interesting exception to this rule [7], it means that fuzzy results is not possible as a direct generalization (fuzzification) of classical results.

Contrary to fuzzy approach, by using R-relations classical result is generalized direct. This is the consequence of the fact that all fundamental characteristics of classical relations (or classical sets) are preserved in the case of R-relations.

4.2 Preference structures as R-relations

The binary R-relation $R = P \cup I$ is called the large preference R-relation of (P, I, J); R is always reflexive. Conversely, given a reflexive binary relation R in A, we can construct a preference structure on A as follows.

Proposition 3: *Consider a reflexive binary R-relation R in A. The R-triplet (P, I, J) of binary R-relations in A constructed as follows:*
(i) $P = R \cap coR^t$;
(ii) $I = R \cap R^t$;
(iii) $J = coR \cap coR^t$,
is a R-preference structure on A such that $R = P \cup I$.

Proof: Let $\otimes = \{R, R^t\}$ be an alphabet (set of primary R-relations) and $\mathcal{P}(\Omega) = \{\emptyset, R, R^t, \{R, R^t\}\}$ corresponding power set . Then the logical structure of binary R-relation R and R^t are set functions:

$$s(R) : \mathcal{P}(\Omega) \to \{0, 1\}$$

given in the following table:

S	$s(R)(S)$	$s(R^t)(S)$
$\emptyset$	0	0
R	1	0
R^t	0	1
$\{R, R^t\}$	1	1

and logical structures of its negations $s(coR)(S) = 1 - s(R)(S)$; $\quad S \in \mathcal{P}(\Omega)$:

S	$s(coR)(S)$	$s(coR^t)(S)$
$\emptyset$	1	1
R	0	1
R^t	1	0
$\{R, R^t\}$	0	0

and since $I = R \cap R^t$; $\quad P = R \cap coR^t$; $\quad P^t = R^t \cap coR$ and $J = coR \cap coR^t$, logical structures of I, P, P^t and J ,based on structural functionality principle, are:

S	$s(S)(I)$	$s(S)(P^t)$
$\emptyset$	0	0
R	0	0
R^t	0	1
$\{R, R^t\}$	1	0

,

S	$s(S)(P)$	$s(S)(J)$
$\emptyset$	0	1
R	1	0
R^t	0	0
$\{R, R^t\}$	0	0

.

It is clear that $J(a,b)$, $P(a,b)$, $P^t(a,b)$, $I(a,b)$ are basic (atomic) R-relation functions:

$$\begin{aligned} J(a,b) &= \mathbf{bR}(\emptyset)(a,b), \\ P(a,b) &= \mathbf{bR}(R)(a,b), \\ P^t(a,b) &= \mathbf{bR}\left(R^t\right)(a,b), \\ I(a,b) &= \mathbf{bR}\left(\left\{R, R^t\right\}\right)(a,b) \end{aligned}$$

or:

$$\begin{array}{l} J(a,b) = 1 - R(a,b) - R(b,a) + R(a,b) \otimes R(b,a) \\ P(a,b) = R(a,b) - R(a,b) \otimes R(b,a) \\ P^t(a,b) = P(b,a) = R(b,a) - R(a,b) \otimes R(b,a) \\ I(a,b) = R(a,b) \otimes R(b,a) \end{array}.$$

It is clear that:

$$P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1; \quad \forall a, b \in A$$

or:

$$\begin{aligned} \left(P \cup P^t \cup I \cup J\right)(a,b) &= 1, \quad \forall (a,b) \in A^2, \\ P \cup P^t \cup I \cup J &= A^2. \end{aligned}$$

This condition can be written equivalently in eight different ways too as a consequence of the following identities:

$$\begin{aligned} s(co(P \cup I))(S) &= s\left(P^t \cup J\right)(S); \\ s(co(P \cup J))(S) &= s\left(P^t \cup I\right)(S); \\ s\left(co\left(P \cup P^t\right)\right)(S) &= s(I \cup J)(S); \end{aligned}$$

$$
\begin{aligned}
s\left(coP^t \cap coI \cap coJ\right)(S) &= s(P)(S)\,; \\
s(coP \cap coI \cap coJ)(S) &= s\left(P^t\right)(S)\,; \\
s\left(coP \cap coP^t \cap coJ\right)(S) &= s(I)(S)\,; \\
s\left(coP \cap coP^t \cap coI\right)(S) &= s(J)(S)\,; \\
s\left(P \cup P^t \cup I \cup J\right)(S) &= s(A)(S)\,; \\
S &\in \mathcal{P}(\Omega)\,.
\end{aligned}
$$

5 Conclusion

R-set theory is a consistent generalization of classical set theory, contrary to theory of fuzzy sets. The basic difference between real set theory and fuzzy set theory lies in the fact that all laws of classical set algebra are valid in the real set theory. Since R-relations are generalization of relations, they have additional properties and capabilities, too.

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