Distorted generalized pseudo-Laplace type transforms

Endre Pap
Ivana Štajner-Papuga

Department of Mathematics and Informatics, University of Novi Sad
Trg D. Obradovica 4, 21 000 Novi Sad, Serbia and Montenegro
e-mail: pape@eunet.yu

Abstract: One form of generalization of the pseudo-Laplace transform, which is an important notion from pseudo-analysis framework often used in dealing with differential or integral equation, has been given. Its role in theory of probabilistic metric spaces has been presented.

Keywords: pseudo-operations, semiring, pseudo-Laplace transforms, generalized pseudo-convolutions, triangle functions.

1 Introduction

The main interest of this paper is generalization of the well known Laplace transform, transform that has essential role in dealing with differential or integral equations. Generalization presented here has been done in the pseudo-analysis framework, that is, it belongs to a rather new theory that has proved itself as a power full tool in various aspects of mathematics ([5, 6, 9, 10, 13, 14, 16, 17, 18]). Of great importance is also generalization of the exchange formula that transforms pseudo-convolution, generalization of the classical convolution that has applications in probabilistic metric spaces, information theory, fuzzy numbers, optimization, etc., into the pseudo-product.

Since probabilistic metric spaces are oriented on distribution functions for measuring distance, operations with distribution functions known as triangle functions which are considered to be the natural generalization of the usual triangle inequality, are of grate importance for this theory ([3, 4, 8, 20]). It has been established that some classes of triangle functions can be interpreted as generalized pseudo-convolutions of distribution functions ([14]), hence previously mentioned exchange formula puts Laplace type transforms, among others, in the core of the theory of probabilistic metric spaces.

The second section of this paper consists of preliminaries notions. A short overview of basic definitions has been given. Pseudo-Laplace type transforms, namely the distorted generalized $(\max, T)$-Laplace transforms, have been given in Section 3. Also, the corresponding exchange formula and some of the properties of distorted generalized $(\max, T)$-Laplace transforms have been given.

2 Preliminaries

Let $[a,b]$ be closed subinterval of $[-\infty, +\infty]$ (in some cases, semiclosed subintervals) and let $\preceq$ be an total order on $[a,b]$. Structure $([a,b], \oplus, \odot)$ is called *semiring* if following hold:

- $\oplus$ is *pseudo-addition*, i.e., $\oplus : [a,b] \times [a,b] \to [a,b]$ is commutative, non-decreasing (with respect to $\preceq$), associative operation with a zero element denoted by $\mathbf{0}$;
- $\odot$ is *pseudo-multiplication*, i.e., $\odot : [a,b] \times [a,b] \to [a,b]$ is commutative, positively non-decreasing $\left(x \preceq y \quad \text{implies} \quad x \odot z \preceq y \odot z,\ z \in [a,b]_+ = \{x : x \in [a,b],\ \ \mathbf{0} \preceq x\}\right)$, associative operation with a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Of special importance for this paper are pseudo-operations defined on the unite square, i.e., well known *triangular norms and triangular conorms* (see [4]).

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class consists of semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function belong to the second class, and the third class contains semirings with both idempotent operations. More on this structure can be found in [5, 6, 9, 12, 13, 14].

Classical measure's counterpart based on semiring $([a,b], \oplus, \odot)$ is known as $\sigma-\oplus-$*decomposable measure* (see [6, 9, 10, 12, 21]). This set function m maps some $\sigma-$algebra Σ into $[a,b]_+$ and fulfills conditions $m(\emptyset) = \mathbf{0}$ and

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} m(A_i)$$

for all sequences $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from Σ. If $\oplus$ is idempotent operation, condition $m(\emptyset) = \mathbf{0}$ and pairwise disjointness of sets can be omitted.

Construction of *pseudo-integral* on semiring $([a,b], \oplus, \odot)$ with respect to $\sigma-\oplus-$decomposable measure m is similar to the construction of classical Lebesgue integral (see [4, 6, 7, 9, 12]). This integral is denoted with $\int_X^{\oplus} f \odot dm$ for all bounded measurable function $f : X \to [a, b]$.

Another notion necessary for this paper is notion of the generalized pseudo-convolution (see [4, 12, 14, 15]). Let G be subset of $\mathbb{R}$, $*$ a commutative binary operation on $\mathbb{R}$ that fulfills cancellation law and $(G, *)$ semigroup with unit element u. Let $G_+ = \{x | x \in G, x \geq u\}$. The *generalized pseudo-convolution of the first type* of two functions $f : G_+ \to [a, b]$ and $h : G_+ \to [a, b]$ with respect to a σ-$\oplus$-decomposable measure m is mapping $f \star h : G_+ \to [a, b]$ given by

$$f \star h(x) = \int_{G_+^x}^{\oplus} f(u) \odot dm_h(v), \tag{1}$$

where G_+^x is set of all $u, v \in G_+$ such that $u * v = x$, $m_h(A) = \sup_{x \in A} h(x)$ for $\oplus = \max$, $m_h(A) = \inf_{x \in A} h(x)$ for $\oplus = \min$, and if $\oplus$ has an additive generator g, then $dm_h = h \odot d(g^{-1} \circ \lambda)$, where $\lambda = g \circ m$ is the Lebesgue measure (generalized g-convolution of the first type). *Generalized pseudo-convolution of the second type* is obtained when $(G, *)$ is a group and pseudo-integral in (1) is taken over the whole set G (see [14]).

Also, the notion of triangle function will be required. Let $\triangle^+$ be *space of probability distribution functions*, i.e.:

$$\triangle^+ = \{F : \mathbb{R}^+ \to [0, 1] \mid F \text{ is left-continuous, non-decreasing,} \\ F(0) = 0 \text{ and } F(\infty) = 1\}.$$

A *triangle function* is a function $\tau : \triangle^+ \times \triangle^+ \to \triangle^+$ such that for all $F, G, H \in \triangle^+$ following hold: $\tau(F, G) = \tau(G, F)$, $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$, $F \leq G \Rightarrow \tau(F, H) \leq \tau(G, H)$ and $\tau(F, \varepsilon_0) = F$. By *probabilistic metric space*, a triple $(M, \mathcal{F}, \tau)$ where M is non-empty set, a function $\mathcal{F} : M^2 \to \triangle^+$ assigns to each $(p, q) \in M^2$ a probability distribution functions F_{pq}, τ is a *triangle function*, $F_{pp} = \varepsilon_0$, $F_{pq} \neq \varepsilon_0$ for $p \neq q$, $F_{pq} = F_{qp}$ and $F_{pr} \geq \tau(F_{pq}, F_{qr})$, where

$$\varepsilon_0(x) = \begin{cases} 0, & x = 0 \\ 1, & x > 0, \end{cases}$$

The importance of previously described pseudo-analysis's machinery for probabilistic metric spaces can by easily shown by the following examples of some essential triangle functions (see [14]).

Example 1 Let T be left continuous t-norm and let F and H be distribution functions from $\triangle^+$. Generalized pseudo-convolution based on $([0, 1], \max, T)$ for $* = +$, i.e.,

$$F \star H(x) = \sup\{T(F(u), H(v)) \mid v + u = x\},$$

is triangle function denoted with $\tau_T(F,H)(x)$. This triangle function is in the core of well known Menger space.

Now, let L be an binary operation on $[0,+\infty]$ which is non-decreasing in both coordinate, continuous on $[0,+\infty)^2$, commutative, associative, has 0 as identity and fulfills cancellation law. The generalized pseudo-convolution based on $([0,1],\max,T)$ for $*=L$ is

$$F\star H(x)=\sup\{T\,(F(u),H(v))\ \mid L(v,u)=x\}$$

where T is left continuous t-norm and $F,H\in\Delta^+$. This pseudo-convolution is well known triangle function $\tau_{T,L}(F,H)(x)$.

Example 2 For S being continuous triangular conorm, the generalized pseudo-convolution based on $([0,1],\min,S)$ for $*=+$ is

$$F\star H(x)=\inf\{S\,(F(u),H(v))\ \mid v+u=x\}\,.$$

This generalized pseudo-convolution is triangle function $\tau_S(F,H)(x)$.

If $\star$ is the generalized pseudo-convolution based on $([0,1],\min,S)$ for $*=L$ as in the previous example, $F\star H(x)$ is triangle function $\tau_{S,L}(F,H)(x)$, i.e.:

$$F\star H(x)=\inf\{S\,(F(u),H(v))\ \mid L(v,u)=x\}=\tau_{S,L}(F,H)(x).$$

3 Distorted generalized $(\max,T)$-Laplace transform

Transformations of the measurable functions done in the pseudo-Laplace style (see [9, 12, 13]) will be investigated through this section. The more general case, the generalized $(\oplus,\odot)$-Laplace transform, can be found in [16]. Here, we shall consider its modification, so called *distorted generalized* $(\max,T)$*-Laplace transform.*

Let L be previously mentioned binary operation on domain of the probability distribution functions from Δ^+ that is non-decreasing in both coordinate, continuous on $[0,+\infty)^2$, commutative, associative, has 0 as identity and fulfills cancellation law. Since this operation is strict, it can be represented by means of generating function (see [1, 4, 13]). Let $l:[0,\infty]\to[0,1]$ be *multiplicative generator* for binary operation L, that is, let l be continuous, decreasing function such that $l(0)=1$ and

$$L(x,y)=l^{-1}\left(l(x)l(y)\right).$$

We shall consider another binary operation $\diamondsuit : [0,\infty]^2 \to [0,\infty]$ that has to be distributive with respect to L, i.e., $L(x,y)\diamondsuit z = L(x\diamondsuit z, y\diamondsuit z)$. If L is given by multiplicative generator, operation $\diamondsuit$ can have only the following form

$$x\diamondsuit y = l^{-1}\left(\exp\left(-\ln l(x)\ln l(y)\right)\right).$$

Further on, semiring of the first class $([0,1],\max,T)$ where T is an Archimedean t-norm with multiplicative generating function θ is considered. Laplace type transform in the style of Moynihan (see [8]) is given by the following definition.

Definition 3 *Let T be an Archimedean t-norm.* Distorted generalized $(\max,T)$ -Laplace transform *for the semigrup $(\Delta^+,\star)$ is mapping $\mathcal{DL}_{T,L}^{\max}$ defined for $F\in\Delta^+$ as*

$$\mathcal{DL}_{T,L}^{\max}F(z) = \max\left\{0, \sup_{x>0} l(x\diamondsuit z)\theta\left(F(x)\right)\right\}, \qquad z\geq 0.$$

Example 4 For $T=T_P$ and $L=+$, distorted generalized $(\max,T)$ -Laplace transform has the following form that coincides with so called Product-conjugate transform used by Moynihan ([8]):

$$\mathcal{DL}_{T_P,+}^{\max}F(z) = \sup_{x\geq 0} e^{-xz}F(x).$$

Remark 5 *Generalized $(\oplus,\odot)$-Laplace transform* of a measurable function $F:[0,\infty)\to[a,b]$ has been introduced in [16] and it has the following from

$$\mathcal{L}_{\odot,L}^{\oplus}F(z) = \int_{[0,\infty)}^{\oplus} \theta^{(-1)}\left(l(x\diamondsuit z)\right)\odot dm_F(x).$$

If $\oplus=\max$ and $\odot$ is an Archimedean t-norm T given by continuous and increasing generating function $\theta:[0,1]\to[0,1]$ with $\theta(1)=1$ (see [4]), generalized $(\max,T)$ -Laplace transform is mapping $\mathcal{L}_{T,L}^{\max}$ defined for all $F:[0,\infty)\to[0,1]$ as

$$\mathcal{L}_{T,L}^{\max}F(z) = \theta^{(-1)}\left(\sup_{x\geq 0} l(x\diamondsuit z)\theta\left(F(x)\right)\right), \qquad z\geq 0,$$

where $\theta^{(-1)}$ is pseudo-inverse function of θ (see [4]).

As mentioned before, exchange-type formulas are of grate importance. It can be proven (see [16]) that the following type of the pseudo-exchange formula holds:

$$\mathcal{DL}_{T,L}^{\max}\left(F_1\star F_2\star\cdots\star F_n\right)(z)$$
$$=\max\left\{\theta(0), \mathcal{DL}_{T,L}^{\max}F_1(z)\mathcal{DL}_{T,L}^{\max}F_2(z)\cdots\mathcal{DL}_{T,L}^{\max}F_n(z)\right\},$$

or, written with respect to t-norm as a pseudo-product $\odot=T$

$$\theta^{-1}\left(\mathcal{DL}_{T,L}^{\max}\left(F_1\star F_2\star\cdots\star F_n\right)(z)\right)$$

$$= \theta^{-1}\left(\mathcal{DL}_{T,L}^{\max} F_1(z)\right) \odot \theta^{-1}\left(\mathcal{DL}_{T,L}^{\max} F_2(z)\right) \odot \cdots \odot \theta^{-1}\left(\mathcal{DL}_{T,L}^{\max} F_n(z)\right).$$

Another important property of distorted generalized $(\max, T)$ -Laplace transform given by following theorem has been proven in [16].

Theorem 6 *Distorted generalized* $(\max, T_P)$ *-Laplace transform is continuous.*

Since for an Archimedean t-norm T given by multiplicative generator θ holds

$$\mathcal{DL}_{T,L}^{\max} F(z) = \mathcal{DL}_{T_P,L}^{\max} F^*(z),$$

where F^* is function from $\triangle^+$ such that it is equal to zero for $x = 0$ and $\theta \circ F(x)$ for all $x > 0$, as the consequence of the previous theorem for some sequences $(F_n)_{n\in\mathbb{N}}$ from $\triangle^+$ and $F \in \triangle^+$, we have following:

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{implies} \quad \lim_{n\to\infty} \mathcal{DL}_{T,L}^{\max} F_n(z) = \mathcal{DL}_{T,L}^{\max} F(z) \qquad (2)$$

for all continuity points x of function F and all $z > 0$.

Remark 7 By limit $\lim\limits_{n\to\infty} F_n(x) = F(x)$ from (2), a *weak convergence* of sequence $(F_n)_{n\in\mathbb{N}}$ towards F has been considered. That is, $\lim\limits_{n\to\infty} F_n(x) = F(x)$ for all continuity points x of function F. In [16], it has been proved that sequence $(F_n)_{n\in\mathbb{N}}$ converges weakly to F if and only if $\mathcal{L}_l(F_n, F) \to 0$, where $\mathcal{L}_l$ is *generalized modified Levy metric* given by

$$\mathcal{L}_l = \inf\{-\ln l(\delta) \;\mid\; F(x) \le G(L(x,\delta)) - \ln l(\delta)$$
$$\text{and } G(x) \le F(L(x,\delta)) - \ln l(\delta) \text{ for } 0 < x < \delta^*\},$$

for $\delta \diamondsuit \delta^* = \mathbf{1}$.

Presented distorted generalized $(\max, T)$ -Laplace transform has essential role in proving following theorem from [16]. This results considers a wide class of triangle functions and provides an answer to the question whether a non-trivial limit for the sequence $(\mathcal{T}_{T,L}(F_1, F_2, \ldots, F_n))_{n\in\mathbb{N}}$, i.e., if written in the sense of pseudo-analysis, $(F_1 \star F_2 \star \cdots \star F_n)_{n\in\mathbb{N}}$, exists or not. In previous notation, F_i, $i \in \mathbb{N}$, are probability distribution functions and $\star$ is generalized pseudo-convolution based on semiring $([0,1], \max, T)$.

Theorem 8 *Let T be an Archimedean t-norm and let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\triangle^+$ and*

$$G_n = F_1 \star F_2 \star \cdots \star F_n,$$

for $n = 1, 2, \ldots$, where $\star$ is generalized pseudo-convolution based on semiring $([0,1], \max, T)$ for $* = L$. Then the sequence $(G_n)_{n\in\mathbb{N}}$ has a non-trivial limit if and only if there is a sequence of positive numbers $(x_n)_{n\in\mathbb{N}}$ such that

$$\bigsqcup_{n=1}^{\infty} x_n < \infty \quad and \quad \prod_{n=1}^{\infty} F_n(x_n) > 0.$$

Specially, for $L(x,y) = x + y$ and $x\Diamond y = xy$, Moynihan's result from [8] can be obtained.

Remark 9 By haveing a non-trivial limit, in Theorem 8, it has been assumed that sequence $(G_n)_{n\in\mathbb{N}}$ does not converge to $\varepsilon_\infty = \begin{cases} 0, & 0 \le x < \infty \\ 1, & x = \infty \end{cases}$.

4 Conclusion

Generalization of the Laplace transform known as pseudo-Laplace transform (see [11, 13]) has proved itself to be useful in solving some nonlinear equations ([6, 9, 10, 11]) as well as in optimization theory and decision theory ([2]). The main aim of this paper has been to present further steps in this process of generalization that could broaden the area of applications. Another problem concerning this topic that should be addressed in the future is existence of inverse generalized $(\oplus,\odot)$-Laplace transform and its possible applications. Another problem concerning this field of mathematics is problem of characterization of triangle functions. For some classes of triangle functions the answer has been given in [19], however, this question, in general, remains open.

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