# Decision making based on aggregation operators

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*Abstract: There is given a short overview of some basic mathematical properties of aggregation operators (based on [14]) related to decision making. Special attention is taken on compensatory operators and operators based on integrals.*

*Keywords: Aggregation operator, decision making, t-norm, t-conorm, Choquet integral, pseudo-analysis.*

## 1 Introduction

Aggregation operators play important role in many different approaches to decision making [9, 13, 17, 25]. Therefore we will investigate some of their properties. In many systems (specially intelligent) the aggregation of incoming data plays the main role. The aggregation operators form a fundamental part of multi-criteria decision making, engineering design, expert systems, pattern recognition, neural networks, fuzzy controllers, genetic algorithms.

The choice of the aggregation operator depends on the actual application. To obtain a sensible and satisfactory aggregation, any aggregation operator should not be used. To choose satisfactory aggregation operators, we can adopt an axiomatic approach and impose that these operators fulfill some selected properties. These properties can be dictated by the nature of the values to be aggregated, e.g., in some multi-criteria evaluation methods, the aim is to assess a global absolute score to an alternative given a set of partial scores with respect to different criteria. It would be unnatural to give as global score a value which is lower than the lowest partial score, or greater than the highest score, so that only *internal* aggregation operators are allowed. If preference degrees coming from transitive (in some sense) relations are combined, it is natural to require that the result of combination remains transitive. Another example is related to the aggregation of opinions in voting procedures. Since usually the voters are anonymous, the aggregation operator have to be symmetric.

Decision making needs more general mathematical models, which involve also non-additive measures. Previously used additive probability measures could not model some situations as e.g. the Ellsberg Paradox, see [13]. For the non-additive set function (measure) $m$ defined on a $\sigma$-algebra $\Sigma$ of subsets of a set $X$ (for finite $X$ it is usually taken $\Sigma = \mathcal{P}(X)$, the family of all subsets), the difference $m(A \cup B) - m(B)$ depends on $B$ and can be interpreted as the effect of $A$ joining $B$, [13, 26, 32].

First we shall present some basic elements on the mathematical properties of the aggregation operators from the book under preparation [14].

## 2 Definition of the aggregation operator

We make a distinction between aggregation operators having one definite number of arguments and extended aggregation operators defined for all number of arguments.

Throughout we denote by $I$ any nonempty real interval, bounded or not. $I^\circ$ denotes the interior of $I$, that is the corresponding open interval.

Let $n$ be any nonzero natural integer and set $[n] := \{1, \ldots, n\}$.

**Definition 1** *An aggregation operator is a function* $\mathsf{A}^{(n)} : I^n \to \mathbb{R}$.

For instance, the arithmetic mean as an aggregation operator is defined by

$$\mathsf{A}^{(n)}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{1}$$

The integer $n$ represents the number of values to be aggregated. When no confusion can arise, the aggregation operators will simply be written $\mathsf{A}$ instead of $\mathsf{A}^{(n)}$.

A specific case is the aggregation of a singleton, i.e., the unary operator $\mathsf{A}^{(1)} : I \to \mathbb{R}$. For many scientists the aggregation (fusion) of a singleton is not an (true) aggregation, so they propose the convention that $\mathsf{A}^{(1)}(x) = x \quad (x \in I)$. Unless otherwise specified, we will adopt this convention throughout.

**Definition 2** *An extended aggregation operator is a sequence* $(\mathsf{A}^{(n)})_{n \geqslant 1}$*, whose* $n$*th element is an aggregation operator* $\mathsf{A}^{(n)} : I^n \to \mathbb{R}$.

For example, the arithmetic mean as an extended aggregation operator is the sequence $(\mathsf{A}^{(n)})_{n \geqslant 1}$, where $\mathsf{A}^{(n)}$ is defined by (1) for all integer $n \geqslant 1$.

Note that, in a general extended operator, for different $n$ and $m$ the operators $\mathsf{A}^{(n)}$ and $\mathsf{A}^{(m)}$ need to be related. This possible defect of extended aggregation operators will be discussed on appropriate places in the next chapters. Sometimes only partial operators $\mathsf{A}^{(n)}$ will be discussed, depending on the topic.

Of course, an extended aggregation operator is a multidimensional operator, which can be also viewed as a mapping

$$\mathsf{A} : \bigcup_{n \geqslant 1} I^n \to \mathbb{R}.$$

In this case we write $\mathsf{A}^{(n)} = \mathsf{A}|_{I^n}$ for all integer $n \geqslant 1$.

For the illustration and the next use we now give some well-known aggregation operators. In the definitions below, $\mathbf{x}$ will stand for $(x_1, \ldots, x_n)$. The *arithmetic mean* operator $\mathsf{AM}$ is defined by $\mathsf{AM}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$. For any weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} \omega_i = 1$, the *weighted arithmetic mean* operator $\mathsf{WAM}_\omega$ and the *ordered weighted averaging* operator $\mathsf{OWA}_\omega$ associated to $\omega$, are respectively defined by $\mathsf{WAM}_\omega(\mathbf{x}) = \sum_{i=1}^{n} \omega_i\, x_i, \quad \mathsf{OWA}_\omega(\mathbf{x}) = \sum_{i=1}^{n} \omega_i\, x_{(i)}$. For any $k \in [n]$, the *coordinate projection* operator $\mathsf{P}_k$ and the *order statistic* operator $\mathsf{OS}_k$ associated to the $k$th argument, are respectively defined by $\mathsf{P}_k(x) = x_k, \quad \mathsf{OS}_k(x) = x_{(k)}$. The projection of the first and the last coordinates are defined as $\mathsf{P}_F(x) = \mathsf{P}_1(\mathbf{x}) = x_1, \mathsf{P}_L(x) = \mathsf{P}_n(\mathbf{x}) = x_n$. Similarly, the extreme order statistics $x_{(1)}$ and $x_{(n)}$ are respectively the minimum and maximum operators $\mathsf{Min}(\mathbf{x}) = \min(x_1, \ldots, x_n), \quad \mathsf{Max}(\mathbf{x}) = \max(x_1, \ldots, x_n)$. Also, the *median* of an odd number of values $(x_1, \ldots, x_{2k-1})$ is simply defined by $\mathsf{Med}(x_1, \ldots, x_{2k-1}) = x_{(k)}$. The sum and product operators are respectively defined by $\mathsf{\Sigma}(\mathbf{x}) = \sum_{i=1}^{n} x_i, \quad \mathsf{\Pi}(\mathbf{x}) = \prod_{i=1}^{n} x_i$. Consider the Gödel implication $\mathsf{I}_G : [0,1]^2 \to [0,1]$,

$$\mathsf{I}_G(x,y) = \begin{cases} y, & \text{if } x > y\,, \\ 1, & \text{else,} \end{cases}$$

which is defined as a binary operator only. Note that starting from the binary operator $\mathsf{I}_G$ we can define an extended operator in several ways. For $n = 1$, we can put $\mathsf{I}_G(x) = x$.

**Notations:** For any integer $k \geqslant 1$ and any $x \in I$, we set $k \cdot x := x, \ldots, x$ ($k$ times). For any vectors $\mathbf{x}, \mathbf{x}' \in I^n$, we denote by $\mathbf{x}\mathbf{x}'$ the $n$-dimensional vector $(x_1x_1', \ldots, x_nx_n')$ obtained by calculated the product componentwise. The vectors $\mathbf{x} + \mathbf{x}'$, $\mathbf{x} \wedge \mathbf{x}'$, and $\mathbf{x} \vee \mathbf{x}'$ are defined similarly.For any $\mathbf{x} \in I^n$ and any function $\phi : I^n \to \mathbb{R}^n$, we denote by $\phi(\mathbf{x})$ the $n$-dimensional vector $\big(\phi(x_1), \ldots, \phi(x_n)\big)$. If $\phi = (\phi_1, \ldots, \phi_n)$, where $\phi_i : I \to \mathbb{R}$ is any function, then $\phi(\mathbf{x})$ stands for $\big(\phi_1(x_1), \ldots, \phi_n(x_n)\big)$.For any finite or denumerable set $K$, we let $\Pi_K$ denote the set of all permutations on $K$.Given a vector $(x_1, \ldots, x_n)$ and a permutation $\sigma \in \Pi_{[n]}$, the notation $[x_1, \ldots, x_n]_\sigma$ means $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$, that is, the permutation $\sigma$ of the indices.

# 3 Basic mathematical properties

In the present section we introduce basic and standard properties often required for aggregation. For example, increasing monotonicity is an indispensable condition for operators used to aggregate preferences. Idempotency is necessary when the aggregated evaluation plays the role of a typical value, etc.

## 3.1 Symmetry

The first property we consider is *symmetry*, also called commutativity, neutrality, or anonymity. The standard commutativity of binary operators $x*y = y*x$, well known in Algebra, can be easily generalized for $n$-ary aggregation operators, with $n > 2$, as follows.

**Definition 3** *$\mathsf{A} : I^n \to \mathbb{R}$ is a symmetric operator if*

$$\mathsf{A}(\mathbf{x}) = \mathsf{A}([\mathbf{x}]_\sigma) \qquad (\mathbf{x} \in I^n,\ \sigma \in \Pi_{[n]}).$$

The symmetry property essentially means that the aggregated value does not depend on the order of the arguments. This is required when combining criteria of equal importance or anonymous expert's opinions, e.g., symmetry is more natural in voting procedures than in multicriteria decision making, where criteria usually have different importances.

Many aggregation operators introduced till now are symmetric. For example, $\mathsf{AM}$, $\mathsf{GM}$, $\mathsf{OWA}_\omega$ are symmetric operators. Prominent examples of nonsymmetric aggregation operators are weighted arithmetic means $\mathsf{WAM}_\omega$.

The following result, well-known in group theory, shows that the symmetry property can be checked with only two equalities.

**Proposition 4** *$\mathsf{A} : I^n \to \mathbb{R}$ is a symmetric operator if and only if, for all $\mathbf{x} \in I^n$, we have*

*i)* $\mathsf{A}(x_2, x_1, x_3, \ldots, x_n) = \mathsf{A}(x_1, x_2, x_3, \ldots, x_n)$,

*ii)* $\mathsf{A}(x_2, x_3, \ldots, x_n, x_1) = \mathsf{A}(x_1, x_2, x_3, \ldots, x_n)$.

In situations when judges, criteria, or individual opinions are not equally important, the symmetry property must be omitted. There are some attempts how to incorporate weights into symmetric operators. Conversely, a nonsymmetric operator $\mathsf{A}(x)$ can always be symmetrized by replacing it with $\mathsf{A}(x_{(1)}, \ldots, x_{(n)})$. Thus, according to this process a weighted arithmetic mean $\mathsf{WAM}_\omega$ gives rise to the corresponding ordered averaging operator $\mathsf{OWA}_\omega$.

## 3.2 Continuity

We now consider the classical property of *continuity*.

**Definition 5** *$\mathsf{A} : I^n \to \mathbb{R}$ is a continuous operator if it is a continuous function in the usual sense.*

The continuity property is required when we want the operator to present no chaotic reaction to any small change of the arguments. For example, when the inputs represent approximate readings we expect that the presence of any small error does not cause a big error in the output.

For nondecreasing operators, continuity is equivalent to the following *intermediate value* property.

**Definition 6** *A nondecreasing operator $\mathsf{A} : I^n \to \mathbb{R}$ has the intermediate value property if, for all $\mathbf{x}, \mathbf{y} \in I^n$ such that $\mathbf{x} \leqslant \mathbf{y}$ and all $c \in [\mathsf{A}(\mathbf{x}), \mathsf{A}(\mathbf{y})]$, there is $\mathbf{z} \in I^n$, with $\mathbf{x} \leqslant \mathbf{z} \leqslant \mathbf{y}$, such that $\mathsf{A}(\mathbf{z}) = c$.*

An important analytical property of functions of $n$ variables allowing us to estimate the error when dealing with imprecise input data is the classical Lipschitz property. Recall that an aggregation operator $\mathsf{A} : I^n \to \mathbb{R}$ fulfills the *Lipschitz property* with constant $c \in ]0, \infty[$ ($\mathsf{A}$ is $c$-Lipschitz for short) if, for any $\mathbf{x}, \mathbf{y} \in I^n$, we have

$$|\mathsf{A}(\mathbf{x}) - \mathsf{A}(\mathbf{y})| \leqslant c \sum_{i=1}^{n} |x_i - y_i|.$$

Clearly, the Lipschitz property (with arbitrary $c$) ensures continuity but not vice-versa.

Within the already introduced aggregation operators, the operator $\mathsf{I}_G$ is an example non-continuous operator. Operators $\mathsf{AM}$, $\mathsf{\Pi}$, $\mathsf{Min}$, $\mathsf{Max}$, $\mathsf{P}_F$, $\mathsf{P}_L$ are continuous aggregation operators which all also fulfill the Lipschitz property. As an example of a continuous aggregation operator which is not Lipschitz for any $c \in ]0, \infty[$, we recall the geometric mean $\mathsf{GM}$,

$$\mathsf{GM}(\mathbf{x}) = \Big(\prod_{i=1}^{n} x_i\Big)^{1/n}.$$

There are many important aggregation operators related to the Lipschitz property, see [14]. There also some other types of continuity as lower semi-continuouty upper semi-continuouty.

### 3.3 Monotonicity

Let us consider the following monotonicity properties: *nondecreasing monotonicity*, *strict increasing monotonicity*, and *unanimous increasing monotonicity*.

**Definition 7** *The operator* $\mathsf{A} : I^n \to \mathbb{R}$ *is nondecreasing (in each argument) if, for any* $\mathbf{x}, \mathbf{x}' \in I^n$,

$$x_i \leqslant x'_i \;\; \forall i \in [n] \quad \Rightarrow \quad \mathsf{A}(\mathbf{x}) \leqslant \mathsf{A}(\mathbf{x}').$$

Note also that among the operators we have already discussed, the Gödel implication $\mathsf{I}_G$ is a binary operator that is not nondecreasing in both arguments. In fact, it is nondecreasing in the second argument and nonincreasing in the first argument.

**Definition 8** *The operator* $\mathsf{A} : I^n \to \mathbb{R}$ *is* strictly increasing *(in each argument) if, for any* $\mathbf{x}, \mathbf{x}' \in I^n$,

$$x_i \leqslant x'_i \;\; \forall i \in [n] \; \textit{and} \; \mathbf{x} \neq \mathbf{x}' \quad \Rightarrow \quad \mathsf{A}(\mathbf{x}) < \mathsf{A}(\mathbf{x}').$$

A nondecreasing aggregation operator presents a nonnegative response to any increase of the arguments. In other terms, increasing any input value cannot decrease the output value. This operator is strictly increasing if, moreover, it presents a positive reaction to any increase of at least one input value.

Note that strict monotonicity implies nondecreasing monotonicity trivially. It also implies *cancellativity*, which means that if $\mathsf{A}(\mathbf{x}) = \mathsf{A}(\mathbf{x}')$ and $x_i = x'_i$ for all $[n]\backslash\{i_0\}$, then $x_{i_0} = x'_{i_0}$. The converse is not true, unless $\mathsf{A}$ is nondecreasing.

Evidently, all these forms of monotonicity are related to the Cartesian partial order, when two input systems $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_m)$ are comparable, that is, only if $n = m$ and $x_i \leqslant x'_i$ for all $i \in [n]$ (or $x_i \geqslant x'_i$). There are alternative approaches to the monotonicity of aggregation operators, and then the corresponding aggregation operators.

### 3.4 Idempotency

In Algebra, we say that $x$ is an idempotent element with respect to a binary operation $*$ if $x * x = x$. This algebraic property can be extended to $n$-ary operators, thus defining the *idempotency* property for aggregation operators. Also called unanimity, agreement, or reflexivity, this property means that if all $x_i$ are identical, $\mathsf{A}(x_1, \ldots, x_n)$ restitutes the common value.

**Definition 9** $\mathsf{A} : I^n \to \mathbb{R}$ *is an idempotent operator if*

$$\mathsf{A}(n \cdot x) = x \qquad (x \in I).$$

Idempotency is in some areas supposed to be a genuine property of aggregation operators, e.g., in multi-criteria decision making [9], where it is commonly accepted that if all criteria are satisfied at the same degree $x$ then also the global score should be $x$.

Now, it is evident that $\mathsf{AM}$, $\mathsf{WAM}_\omega$, $\mathsf{OWA}_\omega$, $\mathsf{Min}$, $\mathsf{Max}$, and $\mathsf{Med}$ are idempotent operators, while $\mathsf{\Sigma}$ and $\mathsf{\Pi}$ are not.

**Definition 10** *An element $x \in I$ is an idempotent for $\mathsf{A} : I^n \to \mathbb{R}$ if $\mathsf{A}(n \cdot x) = x$.*

In $[0,1]^n$ the product $\mathsf{\Pi}$ has no other idempotent elements than the extreme elements 0 and 1. As an example of an operator in $[0,1]^n$ which is not idempotent but has a non-extreme idempotent element, take an arbitrarily chosen element $c \in ]0,1[$ and define the aggregation operator $\mathsf{A}_{\{c\}} : [0,1]^n \to [0,1]$ as follows:

$$\mathsf{A}_{\{c\}}(x_1, \ldots, x_n) = \max\left(0, \min\left(1, c + \sum_{i=1}^{n}(x_i - c)\right)\right).$$

By means of a straightforward computation it is easy to see that the only idempotent elements for $\mathsf{A}_{\{c\}}$ are 0, 1, and $c$.

An operator $\mathsf{A} : I^n \to I$ such that any $x \in \operatorname{ran}(A)$ is idempotent for $\mathsf{A}$ will be called *range-idempotent*. This property will be useful when we will introduce the decomposability property.

**Definition 11** $\mathsf{A} : I^n \to I$ *is a range-idempotent operator if*

$$\mathsf{A}\big(n \cdot \mathsf{A}(x_1, \ldots, x_n)\big) = \mathsf{A}(x_1, \ldots, x_n) \qquad (\mathbf{x} \in I^n).$$

# 4 Location in the real line

Very often, aggregation operators are divided into three classes, each possessing very distinct behavior: *conjunctive* operators, *disjunctive* operators and *internal* operators.

**Definition 12** $\mathsf{A} : I^n \to \mathbb{R}$ *is conjunctive if*

$$\mathsf{A}(\mathbf{x}) \leqslant \min x_i \qquad (\mathbf{x} \in I^n).$$

Conjunctive operators combine values as if they were related by a logical "and" operator. That is, the result of combination can be high only if all the values are high. $t$-norms are suitable functions (defined on $[0,1]^n$) for doing conjunctive aggregation. However, they generally do not satisfy properties which are often requested for multicriteria aggregation, such as idempotence, scale invariance, etc.

**Definition 13** $\mathsf{A} : I^n \to \mathbb{R}$ *is disjunctive if*

$$\mathsf{A}(\mathbf{x}) \geqslant \max x_i \qquad (\mathbf{x} \in I^n).$$

Disjunctive operators combine values as an "or" operator, so that the result of combination is high if at least one value is high. Such operators are, in this sense, dual of conjunctive operators. The most common disjunctive operators are $t$-conorms (defined on $[0,1]^n$). As $t$-norms, $t$-conorms do not possess suitable properties for criteria aggregation.

**Definition 14** $\mathsf{A} : I^n \to \mathbb{R}$ *is internal if*

$$\min x_i \leqslant \mathsf{A}(\mathbf{x}) \leqslant \max x_i \qquad (\mathbf{x} \in I^n).$$

Between conjunctive and disjunctive operators, there is room for a third category, namely *internal* operators. They are located between min and max, which are the bounds of the $t$-norm and $t$-conorm families.

The most often encountered functions in the literature on aggregation are means or averaging functions, such as the weighted arithmetic means. Cauchy considered the *mean* of $n$ independent variables $x_1, \ldots, x_n$ as a function $\mathsf{A}(x_1, \ldots, x_n)$ which should be internal to the set of $x_i$ values. Thus, according to Cauchy, a mean is merely an internal operator.

In multicriteria decision aid, these operators are also called *compensative* operators. In fact, in this kind of operators, a bad (resp. good) score on one criterion can be compensated by a good (resp. bad) one on another criterion, so that the result of the aggregation will be medium.

# 5 Cluster based properties

The properties we will focus on in this section concern the "clustering" character of the aggregation operators. That is to say, we assume that it is possible to partition the set of the arguments into disjoint subgroups, build the partial aggregation for each subgroup and then combine these partial results to get the global value. This condition may take several forms. The strongest one we will present is associativity. Other weaker formulations will also be presented, namely decomposability, autodistributivity, bisymmetry, self-identity.

## 5.1 Associativity

We consider first the associativity functional equation. Associativity of a binary operation $*$ means that $(x * y) * z = x * (y * z)$, so we can write $x * y * z$ unambiguously. If we write this binary operation as a two-place function $f(a, b) = a * b$, then associativity says that $f(f(a, b), c) = f(a, f(b, c))$. For general $f$, this is the *associativity functional equation*.

**Definition 15** $\mathsf{A} : I^2 \to I$ *is associative if, for all* $\mathbf{x} \in I^3$*, we have*

$$\mathsf{A}\big(\mathsf{A}(x_1, x_2), x_3\big) = \mathsf{A}\big(x_1, \mathsf{A}(x_2, x_3)\big). \tag{2}$$

A large number of papers deal with the associativity functional equation (2) even in the field of real numbers. In complete generality its investigation naturally constitutes a principal subject of algebra.

Basically, associativity concerns aggregation of only two arguments. However, it can be extended to any finite number of arguments as follows.

**Definition 16** $\mathsf{A} : \cup_{n \geqslant 1} I^n \to I$ *is associative if* $\mathsf{A}(x) = x$ *for all* $x \in I$ *and if*

$$\mathsf{A}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \mathsf{A}\big(\mathsf{A}(x_1, \ldots, x_k), \mathsf{A}(x_{k+1}, \ldots, x_n)\big) \tag{3}$$

*for all integers* $0 \leqslant k \leqslant n$*, with* $n \geqslant 1$*, and all* $\mathbf{x} \in I^n$*.*

For practical purposes we can start with the aggregation procedure before knowing all inputs to be aggregated. New (additional) input data are then simply aggregated with the current aggregated output.

As examples of associative operators recall $\mathsf{Min}$, $\mathsf{Max}$, $\Sigma$, $\Pi$, $\mathsf{P}_F$, $\mathsf{P}_L$. Operators like $\mathsf{AM}$ and $\mathsf{GM}$ are not associative.

In fact, associativity is a very strong and rather restrictive property, especially together with continuity. Therefore sometimes some modifications of associativity preserving its advantages (from the computational point of view) and extending the freedom in the choice of $\mathsf{A}^{(n)}$, $n > 2$, are introduced. If two (or another number) associative operators $\mathsf{B}$, $\mathsf{C}$ and functions $f$, $g$, $h$ are given such that $\mathsf{A} = f(\mathsf{B} \circ g, \mathsf{C} \circ h)$, i.e.,

$$\mathsf{A}(x_1, \ldots, x_n) = f\big(\mathsf{B}(g(x_1), \ldots, g(x_n)), \mathsf{C}(h(x_1), \ldots, h(x_n))\big),$$

then $\mathsf{A}$ is called a *quasi-associative operator*.

## 5.2 Decomposability

It can be easily verified that the arithmetic mean as an extended operator does not solve the associativity equation (2). So, it seems interesting to know whether there exists a functional equation, similar to associativity, which can be solved by the arithmetic mean, or even by other means such as the geometric mean, the quadratic mean, etc.

On this subject, an acceptable equation, called *associativity of means*, has been proposed for symmetric extended operators and can be formulated as follows

$$\mathsf{A}(x_1,\ldots,x_k,x_{k+1},\ldots,x_n)=\mathsf{A}\big(k\cdot\mathsf{A}(x_1,\ldots,x_k),x_{k+1},\ldots,x_n\big)$$

for all integers $0\leqslant k\leqslant n$, with $n\geqslant 1$.

When symmetry is not assumed, it is necessary to rewrite this property in such a way that the first variables are not privileged. We then consider the following definition.

**Definition 17** $\mathsf{A}:\cup_{n\geqslant 1}I^n\to I$ *is decomposable if* $\mathsf{A}(x)=x$ *for all* $x\in I$ *and if*

$$\mathsf{A}(x_1,\ldots,x_k,x_{k+1},\ldots,x_n)=\mathsf{A}\big(k\cdot\mathsf{A}(x_1,\ldots,x_k),(n-k)\cdot\mathsf{A}(x_{k+1},\ldots,x_n)\big)$$

*for all integers* $0\leqslant k\leqslant n$*, with* $n\geqslant 1$*, and all* $\mathbf{x}\in I^n$*.*

By considering $k=0$ (or $k=n$), we see that any decomposable operator is range-idempotent. It follows that decomposability means that each element of any subset of consecutive elements from $\mathbf{x}\in I^n$ can be replaced with their partial aggregation without changing the global aggregation.

Decomposability also implies that the global aggregation does not change when altering some consecutive values without modifying their partial aggregation. For example,

$$\mathsf{A}(x_2,x_3)=\mathsf{A}(x_2',x_3')\quad\Rightarrow\quad\mathsf{A}(x_1,x_2,x_3,x_4,x_5)=\mathsf{A}(x_1,x_2',x_3',x_4,x_5).$$

It is easy to see that, under range-idempotency, this latter property implies decomposability.

## 5.3 Bisymmetry and related properties

Let us consider the *bisymmetry* property, also called mediality.

**Definition 18** $\mathsf{A}:I^2\to I$ *is bisymmetric if for all* $\mathbf{x}\in I^4$*, we have*

$$\mathsf{A}\big(\mathsf{A}(x_1,x_2),\mathsf{A}(x_3,x_4)\big)=\mathsf{A}\big(\mathsf{A}(x_1,x_3),\mathsf{A}(x_2,x_4)\big).$$

The bisymmetry property is very easy to handle and has been investigated from the algebraic point of view by using it mostly in structures without the property of associativity — in a certain respect, it has been used as a substitute for associativity and also for symmetry.

For $n$ arguments, bisymmetry takes the following form.

**Definition 19** $\mathsf{A} : I^n \to I$ *is bisymmetric if*

$$\mathsf{A}\big(\mathsf{A}(x_{11},\ldots,x_{1n}),\ldots,\mathsf{A}(x_{n1},\ldots,x_{nn})\big) = \mathsf{A}\big(\mathsf{A}(x_{11},\ldots,x_{n1}),\ldots,\mathsf{A}(x_{1n},\ldots,x_{nn})\big)$$

*for all square matrices*

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in I^{n\times n}.$$

Bisymmetry expresses that aggregation of all the elements of any square matrix can be performed first on the rows, then on the columns, or conversely. There are further extensions of such type of operators.

## 5.4 Self-identity

**Definition 20** $\mathsf{A} : \cup_{n\geqslant 1} I^n \to I$ *is a self-identity extended aggregation operator if* $\mathsf{A}(x) = x$ *for all* $x \in I$*, and* $\mathsf{A}\big(x_1,\ldots,x_n,\mathsf{A}(x_1,\ldots,x_n)\big) = \mathsf{A}(x_1,\ldots,x_n)$ *for all integer* $n \geqslant 1$ *and all* $\mathbf{x} \in I^n$*.*

Thus we see that, in the case of self-identity extended operators, adding an element equal to the already established value does not change the aggregation value. This property generalizes the next well known feature of the arithmetic mean. If a sample of inputs $x_1,\ldots,x_n$ is given and $\bar{x}$ is the corresponding arithmetic mean, then adding new additional inputs all equal to $\bar{x}$ will not influence the final arithmetic mean.

In this definition, the last argument is privileged. This can make sense in some situations, even when symmetry is not assumed. For instance, consider a situation in which the arguments are temporal in nature, in this case $x_i$ indicates the $i$th observed reading. In situations in which we feel that the basic underlying process generating the readings is changing we may desire to give more emphasis to the later readings rather than to the former ones.

## 5.5 Invariance properties

Depending on the kind of scale which is used, allowed operations on values are restricted. For example, aggregation on ordinal scales should be limited to operations involving comparisons only, such as medians and order statistics, while linear operations are allowed on interval scales.

To be precise, a *scale of measurement* is a mapping that assigns real numbers to objects being measured. Stevens defined the *scale type* of a scale by giving a class of *admissible transformations*, transformations that lead from one acceptable scale to another.

## 6 Further properties

Some other specific properties of aggregation operators, not mentioned in previous sections, have been investigated in the area of aggregation operators. We briefly recall some of them.

The neutral element is again a well-known notion coming from the area of binary operations. This idea is the background of the general definition.

**Definition 21** *Let* $\mathsf{A} : \cup_{n \geqslant 1} I^n \to I$ *be an aggregation operator. An element* $e \in I$ *is called a neutral element of* $\mathsf{A}$ *if, for any* $i \in [n]$ *and any* $\mathbf{x} \in I^n$ *such that* $x_i = e$*, then*

$$\mathsf{A}(x_1, \ldots, x_n) = \mathsf{A}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

So the neutral element can be omitted from aggregation inputs without influencing the final output. In multi-criteria decision making, assigning a score equal to the neutral element (if it exists) to some criterion means that only the other criteria fulfillments are decisive for the global evaluation.

**Definition 22** $\mathsf{A} : I^n \to \mathbb{R}$ *is additive if*

$$\mathsf{A}(\mathbf{x} + \mathbf{x}') = \mathsf{A}(\mathbf{x}) + \mathsf{A}(\mathbf{x}')$$

*for all* $\mathbf{x}, \mathbf{x}' \in I^n$ *such that* $\mathbf{x} + \mathbf{x}' \in I^n$.

**Definition 23** $\mathsf{A} : I^n \to \mathbb{R}$ *is minitive if*

$$\mathsf{A}(\mathbf{x} \wedge \mathbf{x}') = \mathsf{A}(\mathbf{x}) \wedge \mathsf{A}(\mathbf{x}')$$

*for all* $\mathbf{x}, \mathbf{x}' \in I^n$.

**Definition 24** $\mathsf{A} : I^n \to \mathbb{R}$ *is maxitive if*

$$\mathsf{A}(\mathbf{x} \vee \mathbf{x}') = \mathsf{A}(\mathbf{x}) \vee \mathsf{A}(\mathbf{x}')$$

*for all* $\mathbf{x}, \mathbf{x}' \in I^n$.

We now present the concept of *comonotonicity*. In the context we are interested in it is defined as follows.

**Definition 25** *Two vectors* $\mathbf{x}, \mathbf{x}' \in E^n$ *are said to be comonotonic if there exists a permutation* $\pi \in \Pi_{[n]}$ *such that*

$$x_{\pi(1)} \leq \cdots \leq x_{\pi(n)} \quad \text{and} \quad x'_{\pi(1)} \leq \cdots \leq x'_{\pi(n)}.$$

Thus $\pi$ orders the components of $\mathbf{x}$ and $\mathbf{x}'$ simultaneously. Another way to say that $\mathbf{x}$ and $\mathbf{x}'$ are comonotonic is that $(x_i - x_j)(x'_i - x'_j) \geq 0$ for every $i, j \in [n]$. Thus if $x_i < x_j$ for some $i, j$ then $x'_i \leq x'_j$.

**Definition 26** $\mathsf{A} : I^n \to \mathbb{R}$ *is comonotonic additive if*

$$\mathsf{A}(\mathbf{x} + \mathbf{x}') = \mathsf{A}(\mathbf{x}) + \mathsf{A}(\mathbf{x}')$$

*for all comonotonic vectors* $\mathbf{x}, \mathbf{x}' \in I^n$ *such that* $\mathbf{x} + \mathbf{x}' \in I^n$.

# 7 Compensatory operators and aggregation operators based on integrals

From the application point of view, there exist suggestions to use the special aggregation operators, so-called *compensatory operators* in order to model intersection and union in many-valued logic. The main goal of compensatory operators is to model an aggregation of incoming values. If two values are aggregated by a $t$-norm then there is no compensation between low and high values. On the other hand, a $t$-conorm based aggregation provides the full compensation. None of the above cases covers the real decision making. To avoid such inaccuracies, [33] suggested two kinds of so-called compensatory operators. The first of them was $\gamma$-operator, $\Gamma_\gamma : \cup_{n \in \mathbb{N}} [0,1]^n \to [0,1],\ \gamma \in [0,1],\ n \geq 2$

$$\Gamma_\gamma(x_1, \ldots, x_n) = (\prod_{i=1}^{n} x_i)^{1-\gamma}(1 - \prod_{i=1}^{n}(1 - x_i))^\gamma.$$

Here parameter $\gamma$ indicates the degree of compensation. Note that $\gamma$-operators are a special class of exponential compensatory operators [17]. For a given $t$-norm $T$, $t$-conorm $S$ (not necessarily dual to $T$) and parameter $\gamma$ indicating the degree of compensation, the exponential compensatory operator $E_{T,S,\gamma}$ : $[0,1]^n \to [0,1],\ \ n \geq 2$, is defined by

$$E_{T,S,\gamma}(x_1, \ldots, x_n) = (T(x_1, \ldots, x_n))^{1-\gamma}(S(x_1, \ldots, x_n))^\gamma.$$

It is obvious that $\gamma$-operator is based an $T_\mathbf{P}\ S_\mathbf{P},\ \Gamma_\gamma = E_{T_\mathbf{P}, S_\mathbf{P}, \gamma}$. Further note that $E_{T,S,\gamma}$ is a logarithmic convex combination of $T$ and $S$ and up to the case when $\gamma \in \{0, 1\}$ it is non-associative.Another class of compensatory operators proposed in [33] are so-called convex-linear compensatory operators.

We have proposed an *associative class of compensatory operators* in [16]. The degree of compensation is ruled by two parameters, namely by the neutral element $e$ and the compensation factor $k$. Let $T$ be a given strict $t$-norm with additive generator $t, t(\frac{1}{2}) = 1$, and let $S$ be a given strict $t$-conorm with an additive generator $s, s(\frac{1}{2}) = 1$. For a given $e \in\ ]0,1[\,,\ k \in\ ]0, +\infty[\,$, we define an associative compensatory operator

$$C(T, S, e, k) = C : [0,1]^2 \setminus \{(0,1), (1,0)\} \to [0,1]$$

by

$$C(x,y) = h^{-1}(h(x) + h(y)),$$

where $h : [0,1] \to [-\infty, +\infty]$ is a strictly increasing bijection such that

$$h(x) = \begin{cases} kt(\frac{x}{e}) & \text{if } x \in [0,e] \\ s(\frac{x-e}{1-e}) & \text{if } x \in ]e,1]. \end{cases}$$

Note that on the square $[0,e]^2$, $C$ coincides with the $t$-norm $T_e = (< 0, e, T >)$ (the ordinal sum, see [17]). On the square $[e,1]^2$, $C$ coincide with the $t$-conorm $S_e = (< e, 1, S >)$. On the remainder of its domain, it is $T_{\mathbf{M}} < C < S_{\mathbf{M}}$, and note that small values of parameter $k$ increase the values of $C$ (limitedly to $S_{\mathbf{M}}$) while the large values of $k$ decrease the values of $C$ (limitedly to $T_{\mathbf{M}}$). These type of compensatory operators are special type of uni-norms.

The basic idea of any integral is to aggregate the values of some function on a given universe (inputs) into a single value (output). The correspondence between the special aggregation operators and the special types of integrals was studied, e.g., in [11, 13, 6, 9].

Let $(m_n)_{n\in\mathbb{N}}$ be a system of fuzzy measures $m_n : \mathcal{P}(X_n) \to [0,1], X_n = \{1, \cdots, n\}, m_n(X_n) = 1$. Then the operator $\mathsf{A} : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$, defined by

$$\mathsf{A}(x_1, \ldots, x_n) = (C) \int_{X_n} f dm_n,$$

where the right-hand side is a Choquet integral of the function $f : X_n \to [0,1], f(i) = x_i, i = 1, \cdots, n$, with respect to the fuzzy measure $m_n$, is an aggregation operator.

The class of the Choquet integral based aggregation operators corresponds to the idempotent operators stable under increasing linear transformations on the unit interval, which are commonotone additive, i.e., $\mathsf{A} = \varphi(\mathsf{A}(x_1, \ldots, x_n))$ whenever $\varphi : [0,1] \to [0,1], \varphi(x) = ax + b, a > 0$, and

$$\mathsf{A}(x_1 + y_1, \ldots, x_n + y_n) = \mathsf{A}(x_1, \ldots, x_n) + \mathsf{A}(y_1, \ldots, y_n)$$

whenever $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j \in X_n$. Each Choquet integral based aggregation operator $\mathsf{A}$ is continuous. The commutativity of $\mathbf{A}$ depends on the properties of the underlying fuzzy measures and it is equivalent with the property that $m_n(B)$ depends only on $n$ and the cardinality of $B$.

There is need for the investigation of the relation of the aggregation operators with different types of non-additive (fuzzy) integrals. Further investigation on application of the non-additive integrals (Sugeno, Choquet and their generalizations) in the decision (subjective, multi–criteria) theory as an aggregation operator. These integrals have the advantage with respect to other aggregation operators that the non–additivity of the considered measures takes into account the interaction between criteria. For wider applications the identification of the non-additive measure is crucial. At this moment there are many different approaches by statistics, neural networks, genetics algorithms, combinatorial optimization, etc.

There is a mathematical background, which we shall call pseudo–analysis, for treating problems with uncertainty, nonlinearity and optimization in mathematics and soft computing. Namely, instead of the usual plus and/or product structure of real numbers, other operations (pseudo-operations) on extended reals are considered. Certain parts of such mathematical analysis have been developed in analogy with the classical mathematical analysis as for example measure theory, integration, integral operators, convolution, Laplace transform, see books [17, 26, 27].

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