Investigation of the Effect of Time Scaling in a Soft Computing Based Control Using Fractional Order Derivatives

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Abstract:
In this paper the time-scaling properties of an adaptive control based on a novel branch of Computational Cybernetics is investigated in the control of an approximately modeled non-linear system. Adaptive learning of the system's dynamic properties makes it possible to prescribe the desired tracking error relaxation by purely kinematic terms also containing fractional order time-derivatives for which Caputo's definition is used. This definition leads to nonlinear dependence in the length of the cycle time of the controller numerically estimating the time-derivatives. Therefore the division of the cycle-time of the central control unit responsible for dealing with dynamic coupling to smaller intervals available for the local controllers influences the quality of the control as a whole. Simulation results are given to demonstrate the limits of time-resolution that can be applicable in the case of this approach. It is concluded that the method is relatively robust, and that the most important limit is set to the cycle-time of the central unit.

1 Introduction

A new approach for the adaptive control of imprecisely known dynamic systems under unmodeled dynamic interaction with their environment was initiated in [1]. Instead of tuning the supposed analytical model's parameters certain variables of certain simple, and lucid uniform structures are determined by a fast algorithm that finds a certain linear transformation to map a very primitive initial model based expected system-behavior to the observed one. The so obtained „amended model" is step by step updated to trace changes by repeating this corrective mapping in each control cycle. Since no any effort is exerted to identify the possible reasons of the difference between the expected and the observed response, it is referred to as the idea of "Partial System Identification" that is very similar to the main point of the approach applied in various contexts [e.g. 2, 3]. This anticipates the possibility for real-time applications. The conditions of convergence of this approach was investigated in a wider context in [4] based on the modification of the Renormalization Transformation used in many fields of Physics [e.g. 5]. Regarding the appropriate linear transformations several algebraic possibilities were investigated and successfully applied. For instance, the „Generalized Lorentz Group" [6], the „Stretched Orthogonal Group", the "Partially Stretched Orthogonal Transformations" [7], and a special family of the „Symplectic Transformations" [8] can be mentioned.

In the previous investigations the desired trajectory tracking was specified by using purely kinematic terms that corresponded to the classic PID controller determining the 2^{nd} time-derivative of the tracking error as a linear combination of the 0^{th} and 1^{st} derivative of the error plus a small integrating term responsible for removing small and slowly varying residual error that does not give considerable feedback in the previous two terms. This construction can cause a kind of noise-sensitivity due to the "local" nature of the proportional and the derivative terms. It is expected that this situation can be improved if instead of the "local" 2^{nd} order derivative a fractional order derivative of "long memory" is prescribed in the strategy. This approach leads to non-linear time dependence in the control the effect of which is investigated in the sequel. Finally representative simulation results are presented in which the paradigm of the non-linear system to be controlled is a cart to which a double pendulum is attached.

2 Principles of the adaptive control

From purely mathematical point of view the can be formulated as follows. There is given some imperfect model of the system on the basis of which some excitation is calculated to obtain a desired system response $\mathbf{i}^d$ as $\mathbf{e}=\boldsymbol{\varphi}(\mathbf{i}^d)$. The system has its inverse dynamics described by the unknown function $\mathbf{i}^r=\boldsymbol{\psi}(\boldsymbol{\varphi}(\mathbf{i}^d))=\boldsymbol{f}(\mathbf{i}^d)$ and resulting in a realized response $\mathbf{i}^r$ instead of the desired one, $\mathbf{i}^d$. Normally one can obtain information via observation only on the function $\boldsymbol{f}()$ considerably varying in time, and no any possibility exists to directly "manipulate" the nature of this function: only $\mathbf{i}^d$ as the input of $\boldsymbol{f}()$ can be "deformed" to $\mathbf{i}^{d*}$ to achieve and maintain the $\mathbf{i}^d=\boldsymbol{f}(\mathbf{i}^{d*})$ state. [Only the *model function* $\boldsymbol{\varphi}()$ can directly be manipulated.] On the basis of the modification of the method of renormalization widely applied in Physics the following "scaling iteration" was suggested for finding the proper deformation:

$$\mathbf{i_0}; \mathbf{S}_1\mathbf{f}(\mathbf{i_0}) = \mathbf{i_0}; \mathbf{i}_1 = \mathbf{S}_1\mathbf{i_0}; \ldots; \mathbf{S}_n\mathbf{f}(\mathbf{i}_{n-1}) = \mathbf{i_0};$$
$$\mathbf{i}_{n+1} = \mathbf{S}_{n+1}\mathbf{i}_n; \mathbf{S}_n \xrightarrow[n\to\infty]{} \mathbf{I} \tag{1}$$

in which the $\mathbf{S}_n$ matrices denote some linear transformations to be specified later. As it can be seen these matrices map the observed response to the desired one, and the construction of each matrix corresponds to a step in the adaptive control. It is evident that if this series converges to the identity operator just the proper deformation is approached, therefore the controller „learns" the behavior of the observed system by step-by-step amendment and maintenance of the initial model. (The response arrays may contain a „dummy", that is physically not interpreted dimension of constant value, in order to evade the occurrence of the mathematically dubious 0→0, 0→finite, finite→0 cases.) Since (1) does not unambiguously determine the possibly applicable quadratic matrices, we have additional freedom in choosing appropriate ones. The most important points are fast and efficient computation and the ability for remaining as close to the identity transformation as possible. A plausible solution is so placing further blocks around these arrays that the results are easily invertible special matrices belonging to some special Lie Group, in this case to the Symplectic Group. Therefore unique proposition can be given for the $\mathbf{S}_n$ matrices via simple computation. These matrices automatically approach the unit matrix as $\mathbf{f}\to\mathbf{i}_0$, and since they are the elements of a group, their matrix multiplication also belongs to this group, that is we remain within the mathematical frameworks of a simple Lie group the elements of which arbitrarily can approach the unit matrix. In the lack of enough free space, regarding the details we refer to [8].

Since amongst the conditions for which the convergence of the method was proved near-identity transformations were supposed in the perturbation theory, a parameter ξ measuring the „extent of the necessary transformation", a „shape factor" s, and a „regulation factor" λ can be introduced in a linear interpolation with small positive ε_1, ε_2 values as

$$\xi := \frac{\left|\mathbf{f} - \mathbf{i}^d\right|}{\max(|\mathbf{f}|, |\mathbf{i}^d|)}, \quad \lambda = 1 + \varepsilon_1 + (\varepsilon_2 - 1 - \varepsilon_1)\frac{s\xi}{1 + s\xi}, \quad \mathbf{i}^{d*} := \mathbf{f} + \lambda(\mathbf{i}^d - \mathbf{f}) \tag{2}$$

This interpolation reduces the task of the adaptive control in the more critical session and helps to keep the necessary linear transformation in the vicinity of the identity operator.

3 The use of fractional order derivatives

In the case of a normal PID-type controller the desired trajectory reproduction can be prescribed in a purely kinematics based manner. For the second time-derivative of the actual coordinate errors the desired relation can be formulated as:

$$\ddot{\mathbf{e}}^d = -P\mathbf{e} - D\dot{\mathbf{e}} - I\int_0^t \mathbf{e}(t')dt' \tag{3}$$

The main idea of the present modification consists in replacing the local 2nd order derivative by a "global" term having some "memory" as

$$\frac{d^\gamma \mathbf{e}^d}{dt^\gamma} = A\left(-P\mathbf{e} - D\dot{\mathbf{e}} - I\int_0^t \mathbf{e}(t')dt'\right) \tag{4}$$

in which A depends on γ, and the symbol d^γ / dt^γ denotes the fractional order derivative constructed on the basis by Caputo's definition as

$$\frac{d^{\beta+1}}{dt^{\beta+1}}u(t) := \frac{1}{\Gamma(1-\beta)}\int_0^t \left[\frac{d^2u(\tau)}{d\tau^2}\right](t-\tau)^{-\beta}d\tau\ ,\ \beta \in (0,1),\ \gamma := \beta+1 \tag{5}$$

For t>0 (5) physically has the following simple meaning: the full 2nd order derivative in the integrand removes the constant component of the 1st derivative from the signal, and this derivative is "causally reintegrated" by the use of a Green function that has a slowly forgetting nature (the contribution of the far past becomes more and more negligible in it), while its singularity in $\tau=t$ enhances the relative weight of the contribution of the $\tau \cong< t$ instants. Furthermore, the relatively slowly decreasing "tail" of this function also acts as a frequency filter that rejects the high-frequency components of the traditional 2nd derivative. Due to the singularity of the Green function in (5) a common finite-element numerical integration cannot accurately be done. Instead of that, we can suppose that at least $u''(\tau)$ is a relatively slowly varying function of time, therefore it can approximately be treated as a constant during the integration over a small time-interval, while the variation of the Green function can be taken into account accurately. Furthermore, to introduce symmetry against the translation of the signal in time we can omit the very long tail of the Green-function and we can go back in time only to some time t-T instead of 0. The proposed approximation of (5) in this paper was taken as

$$\frac{d^{\beta+1}}{dt^{\beta+1}}u(t) \cong \frac{u''(t)\delta^{-\beta+1}}{\Gamma(2-\beta)} + \sum_{0<s \text{ while } s\delta<T} \frac{\delta^{-\beta+1}\left[(s+1)^{-\beta+1} - s^{-\beta+1}\right]}{\Gamma(2-\beta)}u''(t-s\delta) \tag{6}$$

In the numerical simulations in this paper δ varies from 2.0 to 6 *ms*, while T=10×δ *ms* was chosen. The components of the Green function were stored in an array variable, the u'' values were stored in a shift-register. Equation (6) can trivially be written in the form of

$$\frac{d^{\beta+1}}{dt^{\beta+1}}u_k \cong \sum_{l=0}^{-N} a_l u''_{k+l} \tag{7}$$

that trivially implies that if the conventional 2nd derivative is constant then the appropriate fraction order derivative also is constant, and their ratio is set by the sum of the coefficients $A := \sum_{s=0}^{-N} a_s$ that is taken into account in (4). It is also evident that by prescribing constant fractional order derivative in a discrete time-resolution a causal series of the 2nd order derivatives can be obtained as

$$\frac{d^{\beta+1}}{dt^{\beta+1}}u_{k+1} = a_0 u''_{k+1} + \sum_{l=-1}^{-N} a_l u''_{k+1+l} = \sum_{s=0}^{-N} a_s u''_{k+s} = \frac{d^{\beta+1}}{dt^{\beta+1}}u_k \tag{8}$$

leading to

$$\left(u''_{k+1} - u''_k\right) = -\sum_{s=-1}^{-N} \frac{a_s}{a_0}\left(u''_{k+1+s} - u''_{k+s}\right) \tag{9}$$

It is interesting to see if the "initial condition problem"

$$\frac{d^{\beta+1}}{dt^{\beta+1}}u_{k+s} = \sum_{l=0}^{-N} a_l u''_{k+s+l} \equiv const.\ \ s = 0, 1, \ldots \tag{10}$$

converges to a series of constant 2nd derivatives or results in some divergent series. (In this case the term "initial condition" refers to an N+1 elements long series belonging to s=0.) According to (9) the following estimation can be done:

$$\begin{aligned} \left|u''_{k+1} - u''_k\right| &\leq \sum_{s=-1}^{-N} \left|\frac{a_s}{a_0}\right| \left|u''_{k+1+s} - u''_{k+s}\right| \leq K \times N \times \underset{s=-1,\ldots,-N}{Max}\left|u''_{k+1+s} - u''_{k+s}\right| = \\ &= KN\left|u''_{k+1+s_1} - u''_{k+s_1}\right|, \text{ where } K := \underset{s=-1,\ldots,-N}{Max}\left|\frac{a_s}{a_0}\right|, \ -N \leq s_1 \leq -1, \end{aligned} \tag{11}$$

that is the appropriate maximum is taken at s_1. If KN<1 (11) can recursively applied as

$$\begin{aligned} &\left|u''_{k+1} - u''_k\right| \leq KN\left|u''_{k+1+s_1} - u''_{k+s_1}\right| \leq (KN)^2\left|u''_{k+1+s_1+s_2} - u''_{k+s_1+s_2}\right| \leq \ldots \\ &\leq (KN)^m\left|u''_{k+1+s_1+s_2+\ldots+s_m} - u''_{k+s_1+s_2+\ldots+s_m}\right|, \text{ where} \\ &\underset{z=1,2,\ldots,m}{\forall} \ -N \leq s_z \leq -1 \end{aligned} \tag{12}$$

In (12) sooner or later the maximal difference determined by the initial conditions will be achieved. Since for $k\to\infty$ $m\to\infty$, if KN<1 (9) corresponds to a *Cauchy series* that is convergent in a full metric space, the differences between the elements in the initial condition slowly relax, and the series converges to a constant 2nd derivative.

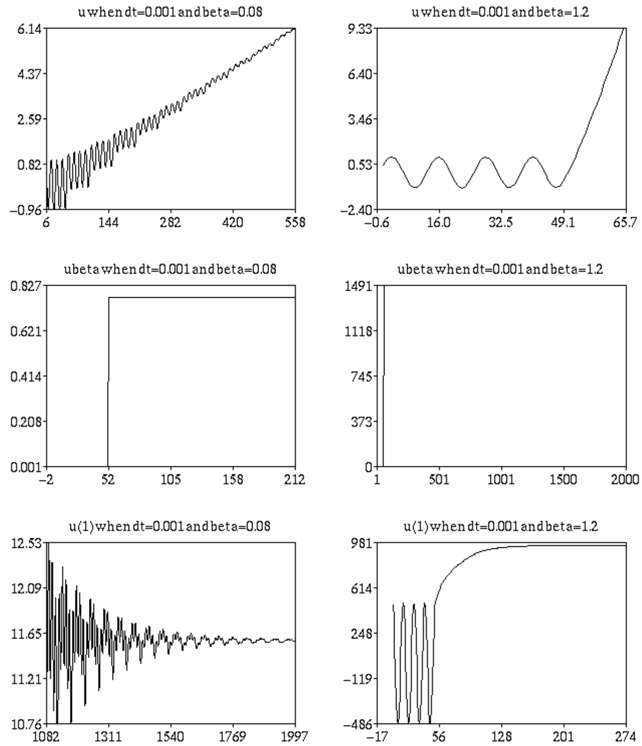

Figure 1. Typical examples for the relaxation of the differences in the 2nd derivatives in the time-series of constant fractional order derivatives obtained from given periodic initial conditions. In contrast to (5) the numerical extension can be calculated for β>1, too. The β=1 case results in the conventional integer order derivative in the approximation. [The connection between the quantities in the figure and the equations: $dt \equiv \delta$ in s units, $ubeta \equiv d^{\beta+1}u/dt^{\beta+1}$, $u(1) \equiv u''$, u stands for u'.]

In Fig. 1 typical examples of the relaxation of the initial conditions can be observed. Increasing β results in more inert behavior. The above results evidently indicate that (4) can be used for smoothing and damping the noise-sensitivity in contrast to the original strategy (3).

4 Simulation results

The system to be controlled was a cart plus a double pendulum system. The cart considered consisted of a body and wheels of negligible momentum and inertia having the overall mass of M [kg]. The pendulums were assembled on the cart by parallel shafts and arms of negligible masses and lengths L_1 and L_2 [m],

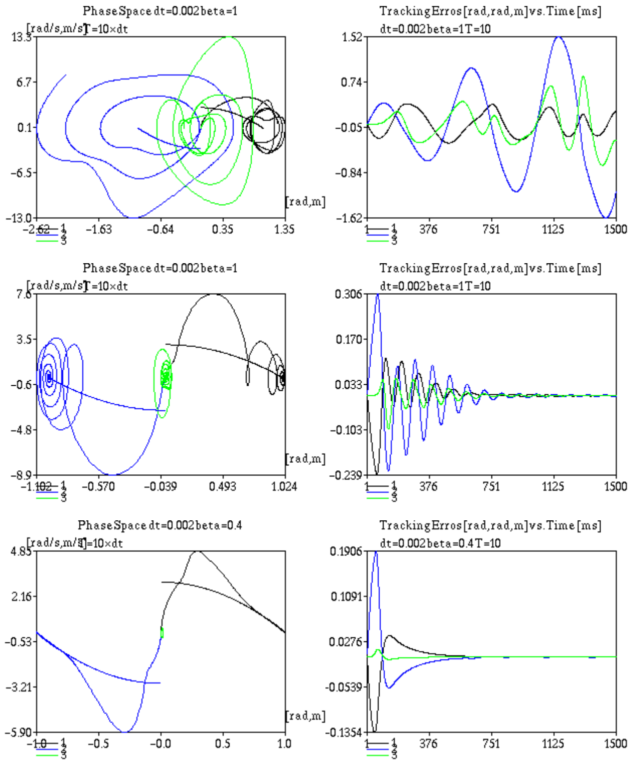

Figure 2. Trajectory tracking improvement caused by adaptivity and by decreasing the order of derivation in the kinematically formulated tracking strategy: tracking in the phase-space (1st column), and in the joint coordinates' space (2nd column); non-adaptive control based on the rough model and integer 2nd derivative (1st row), adaptive control and integer 2nd derivative (2nd row), adaptive control and 1.4th order derivative (3rd row). [*rad/s* vs. *rad* for the rotary joints (black and blue lines) and *m/s* vs. *m* for the linear motion (green lines) in the phase-space, and *rad* and *m* vs. *s* in the 2nd column.]

respectively. At the end of the arms balls of negligible sizes and considerable masses of m_1 and m_2 [*kg*] were attached, respectively. The Euler-Lagrange equations of motion of this system are given as follows:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} m_1 L_1^2 & 0 & -m_1 L_1 \sin q_1 \\ 0 & m_2 L_2^2 & -m_2 L_2 \sin q_2 \\ -m_1 L_1 \sin q_1 & -m_2 L_2 \sin q_2 & \left(M + m_1 + m_2\right) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \\ + \begin{bmatrix} -m_1 L_1 \cos q_1 \dot{q}_1 \dot{q}_3 - m_1 g L_1 \cos q_1 \\ -m_2 L_2 \cos q_2 \dot{q}_2 \dot{q}_3 - m_2 g L_2 \cos q_2 \\ -m_1 L_1 \cos q_1 \dot{q}_1^2 - m_2 L_2 \cos q_2 \dot{q}_2^2 \end{bmatrix} \tag{13}$$

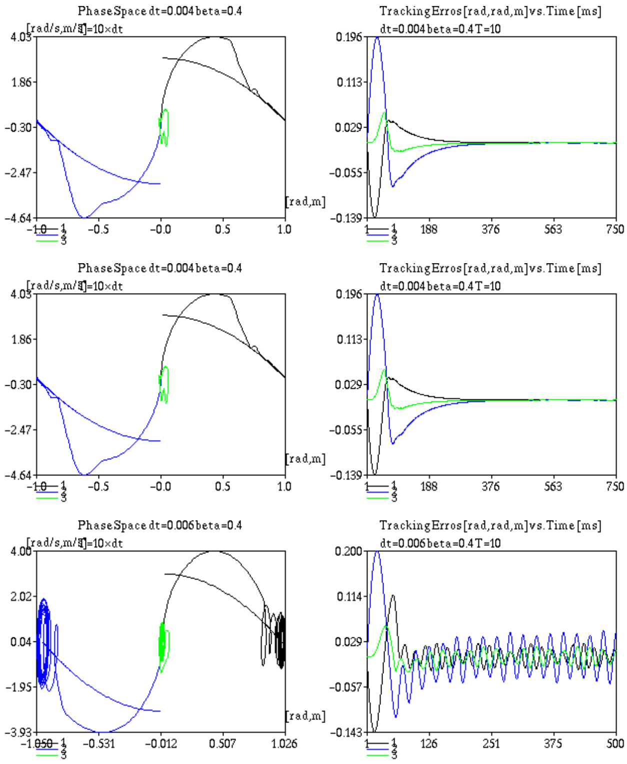

Figure 3. The effect of adding noise of 100 *rad/s*2 and *m/s* to the joint coordinate acceleration measurements and increasing the cycle-time of the controller: 1×2 *ms* (1st row), 1×4 *ms* (2nd row), and 1×6 *ms* (3rd row) [The same units as in Fig. 2.]

respectively. In the simulations L_1=L_2=3 *m*, m_1=20 *kg*, m_2=30 *kg*, *M*=20 *kg* were considered with the supposition of perfect drives that are immediately able to exert the necessary forces and momentums prescribed by the control strategy. The rough dynamic model used consisted of a constant diagonal inertia matrix of the elements <10, 10, 10> and of a constant gravitational and Coriolis term $[10,10,10]^T$.

In Fig. 1 typical simulation results are given for the conventional 2nd order derivative approach in the PID control (for the non-adaptive and the adaptive case), and for the $\gamma=\beta+1=1.4$th order derivative in (4). In each case the cycle-time of the central control was equal to that of the local controllers δ=2 *ms*, and no measurement noise was supposed in the observation of the joint coordinate accelerations. The improvement in the trajectory tracking property is evident due to the application of the adaptive law as well as due to the fractional order derivative in (4).

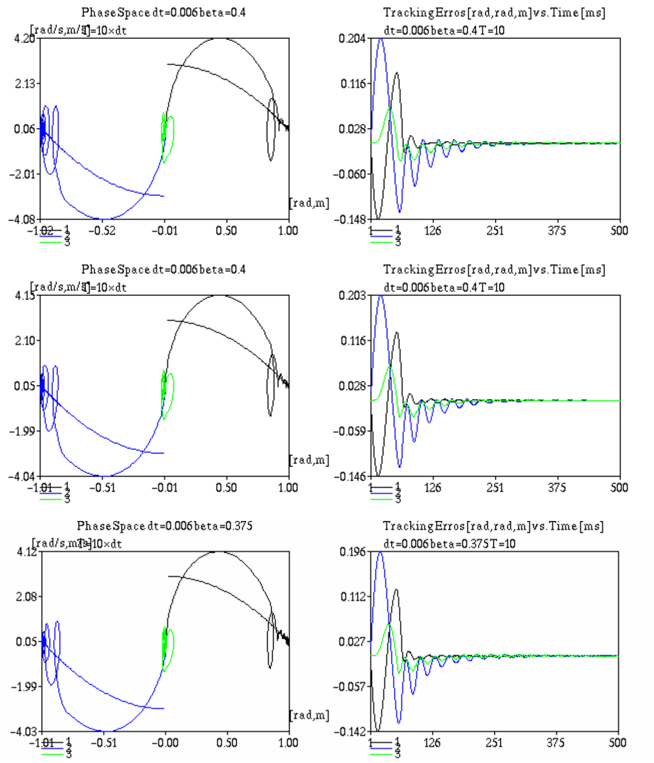

Figure 4. The effect of adding noise of 100 *rad/s*2 and *m/s* to the joint coordinate acceleration measurements and decreasing the cycle-time of the local controllers: 3×2 *ms*, γ=1.4 (1st row), 2×3 *ms*, γ=1.4 (2nd row), and 2×3 *ms*, γ=1.375 (3rd row) [The same units as in Fig. 2.]

In Fig. 3 a random measurement noise of 100 rad/s^2 or 100 m/s^2 was added to the joint coordinate acceleration data, and in each case the cycle-time of the central control was equal to that of the local controllers, but this time interval was increased from 2 to 6 *ms*. It is evident that increasing cycle time at about 6 *ms* means considerable degradation in this dynamic, computed torque like strategy.

Since the approximation (6) is strongly non-linear in the discrete time-resolution δ, various distribution of the computational time between the central and the local controllers was supposed in Fig. 4.

It is evident that the main limitation in this adaptive control is the cycle-time of the central controller, that is the frequency of comparing the rough-model-based expected system-behavior with that of the observed one. However, by comparing the possible resolutions as 1×6 *ms*, 3×2 *ms*, and 2×3 *ms*, the latter seems to be the best.

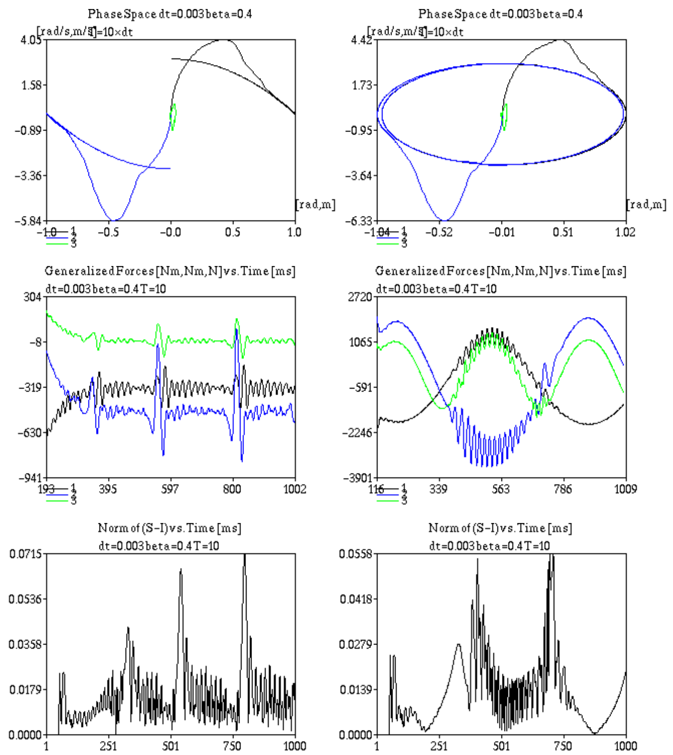

Figure 5. Typical results for the 2×3 *ms*, γ=1.4 case without measurement noise for an asymptotically constant and a periodic nominal trajectory: the phase-space of the nominal and the simulated motion (1st row), zoomed excerpts of the non-transient (i.e. not initial part) of the joint generalized forces exerted by the drives (2nd row), and the norm of the difference between the appropriate "learning Symplectic Matrices" and the unit matrix (non-dimensional).

To reveal the variation of the internal variables of the adaptive controller typical data are given if Fig. 5 for two expected trajectories. As it can be seen the learning transformations are really in the vicinity of the identity operator and the generalized forces show "decent" behavior, too, in the case of the "asymptotic" and the "cyclic" nominal trajectories, too.

5 Conclusions

In this paper the time-scaling properties of an adaptive control based on a novel branch of Computational Cybernetics was investigated in the control of an approximately modeled non-linear system. Adaptive learning of the system's dynamic properties made it possible to prescribe the desired tracking error relaxation by purely kinematic terms also containing fractional order time-derivatives for which a numerical approximation of Caputo's definition was used. This definition leads to nonlinear dependence in the length of the cycle time of the controller numerically estimating the time-derivatives. The effect of the division of the cycle-time of the central control unit responsible for dealing with dynamic coupling to smaller intervals available for the local controllers was investigated via simulation.. Simulation results demonstrated the limits of time-resolution that can be applicable in the case of this approach. It can be concluded that the method is relatively robust, and that the most important limit is set to the cycle-time of the central unit. Finer division of the cycle time of the central unit gives better possibility to estimate the fractional order derivatives at the local level, but due to the necessary memory needed sets a lower limit to this division. At least 2 or 3 *ms* is necessary at the local level otherwise the "remembering" coefficients in the fractional order derivatives become too small and practically local derivatives are taken into account in the control.

6 Acknowledgment

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