Fixed point theorem of Caristi's type

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Abstract: The main purpose of this paper is to give a generalization of the well-known Caristi fixed point theorem in Menger spaces, where t-norm T is of H-type.

Keywords: Fixed point, Menger space, t-norm, probabilistic q-contraction, Cauchy sequence.

1 Introduction

The notion of a probabilistic space is introduced in 1942 by K. Menger. The first idea of K. Menger was to use distribution function instead of nonnegative real numbers as values of the metric. Since then the theory of probabilistic metric spaces has been developed in many directions [5]. Some fixed point results for single-valued and multi-valued mappings in probabilistic metric spaces can be found in [4], [7].

One of the most important results for the fixed point theory in metric space $(M,\, d)$ is the Banach contraction principle.

A mapping $f : M \to M$ is said to be a q-contraction if there exists $q \in [0,\, 1)$ such that

$$d(fx,\, fy) \leq q d(x,\, y)$$

for every $x,\, y \in M$.

Every q-contraction $f : M \to M$ on a complete metric space $(M,\, d)$ has one and only one fixed point.

Sehgal and Bharucha-Reid introduced in [6] the notion of a probabilistic q-contraction ($q \in (0,\, 1)$) in probabilistic metric space.

Definition 1 *Let $(S,\, \mathcal{F})$ be a probabilistic metric space. A mapping $f : S \to S$ is a probabilistic q-contraction if*

$$F_{fp_1,fp_2}(x) \geq F_{p_1,p_2}(\frac{x}{q})$$

for every $p_1,\, p_2 \in S$ *and every* $x \in \mathbb{R}$.

The first fixed-point theorem in probabilistic metric space was proved by Sehgal and Bharucha-Reid in [6].

Theorem 2 *Let* $(S,\, \mathcal{F},\, T_M)$ *be a complete Menger space and* $f : S \to S$ *a probabilistic* q*-contraction. Then there exists a unique fixed point* x *of the mapping* f *and* $x = \lim\limits_{n\to\infty} f^n p$ *for every* $p \in S$.

Banach's contraction principle in metric spaces is a consequence of the well known Caristi fixed point theorem [1], which is one of the most important results in fixed point theory and nonlinear analysis.

Theorem 3 **(Caristi)** *Let* $(M,\, d)$ *be a complete metric space and* $\phi : M \to \mathbf{R}$ *a lower semi-continuous function with a finite lower bound. Let* $f : M \to M$ *be any (not necessarily continuous) function such that*

$$d(x,\, f(x)) \leq \phi(x) - \phi(f(x)) \quad \textit{for} \quad \textit{every } x \in M. \tag{1}$$

Then f *has a fixed point.*

Suppose that $f : M \to M$ is a q-contraction on M. Then $d(fx,\, f^2x) \leq qd(x,\, fx)$ for every $x \in M$ and so

$$d(x,\, fx) - qd(x,\, fx) \leq d(x,\, fx) - d(fx,\, f^2x)$$

for every $x \in M$, which implies that

$$d(x,\, fx) \leq \frac{1}{1-q}d(x,\, fx) - \frac{1}{1-q}d(fx,\, f^2x).$$

This means that for $\phi(x) = \frac{1}{1-q}d(x,\, fx)$ the inequality (1) is satisfied.

In this paper we have proved some fixed point theorem of Caristi's type in Menger space.

2 Preliminaries

Let $\mathcal{D}^+$ be the set of all distribution functions F such that $F(0) = 0$ (F is a non-decreasing, left continuous mapping from $\mathbb{R}$ into $[0, 1]$ such that $\sup\limits_{x\in\mathbb{R}} F(x) = 1$).

Definition 4 [5] *The ordered pair* $(S,\mathcal{F})$ *is said to be a probabilistic metric space if* S *is a nonempty set and* $\mathcal{F}: S\times S\to\mathcal{D}^{+}$ *so that the following conditions are satisfied (where* $\mathcal{F}(p,q)$ *is written by* $F_{p,q}$ *for every* $(p,q)\in S\times S$*):*

1. $F_{p,q}(x)=1$ *for every* $x>0\Leftrightarrow p=q$ $(p,\, q\in S)$.
2. $F_{p,q}=F_{q,p}$ *for every* $p,\, q\in S$.
3. $F_{p,q}(x)=1$ *and* $F_{q,r}(y)=1\Rightarrow F_{p,r}(x+y)=1$ *for* $p,\, q,\, r\in S$ *and* $x,y\in\mathbb{R}^{+}$.

Definition 5 [5] *Recall that a mapping* $T:[0,1]\times[0,1]\to[0,1]$ *is called a triangular norm (a t-norm) if the following conditions are satisfied:*
$T(a,1)=a$ *for every* $a\in[0,1]$ *;* $T(a,b)=T(b,a)$ *for every* $a,b\in[0,1]$;

$$a\geq b,\ c\geq d\Rightarrow\ \ T(a,c)\geq T(b,d)\ \ (a,b,c,d\in[0,1]);$$

$$T(a,T(b,c))=T(T(a,b),c)\ \ (a,b,c\in[0,1]).$$

Example 6 The following are the four basic t-norms :

(i) The minimum t-norm T_M is defined by

$$T_M(x,y)=\min(x,y),$$

(ii) The product t-norm T_P is defined by

$$T_P(x,y)=x\cdot y,$$

(iii) The Lukasiewicz t-norm T_L is defined by

$$T_L(x,y)=\max(x+y-1,0),$$

(iv) The weakest t-norm, the drastic product T_D, is defined by

$$T_D(x,y)=\begin{cases}\min(x,y) & \text{if } \max(x,y)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Definition 7 [5] *A Menger space is an ordered triple* $(S,\mathcal{F},T)$*, where* $(S,\mathcal{F})$ *is a probabilistic metric space,* T *is a* t *-norm and the generalized triangle inequality*

$$F_{p,q}(x+y)\geq T(F_{p,r}(x),F_{q,r}(y))$$

holds for every $p,\, q,\, r\in S$ *and every* $x>0, y>0$.

In [2] a class of t-norms is introduced, which is useful in the fixed point theory in probabilistic metric spaces.

Let T be a t-norm and $T_n : [0,\,1] \to [0,\,1]$ $(n \in \mathbb{N})$ be defined in the following way:

$$T_1(x) = T(x,\,x),\;\; T_{n+1}(x) = T(T_n(x),\,x)\;\; (n \in \mathbb{N},\; x \in [0,\,1]).$$

We say that t-norm T is of H-type if the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$.

Each t-norm T can be extended (by the associativity) in a unique way to an n-ary operation taking for $(x_1, \ldots, x_n) \in [0,1]^n$ the values $T(x_1,\, x_2,\, \ldots, x_n)$, which is defined by

$$\mathbf{T}_{i=1}^{0} x_i = 1,\; \mathbf{T}_{i=1}^{n} x_i = T(\mathbf{T}_{i=1}^{n-1} x_i, x_n).$$

A t-norm T can be extended to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,\,1]$ the value

$$\mathbf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{n} x_i.$$

The sequence $(\mathbf{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below, hence the limit $\mathbf{T}_{i=1}^{\infty} x_i$ exists.

The $(\epsilon, \lambda)-$ topology in S is introduced by the family of neighbourhood of $v \in S$ $\mathcal{U}_v = \{U_v(\epsilon, \lambda)\}_{\epsilon,\lambda)\in\mathbb{R}_+\times(0,1)}$, where

$$U_v(\epsilon, \lambda) = \{u;\; F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If a t-norm T is such that $\sup\limits_{x<1} T(x, x) = 1$, then $\{\mathcal{U}_v\}_{v\in S}$ defines on S a metrizable topology.

Let $(S,\, \mathcal{F})$ be a probabilistic metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in S is a Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0,\,1)$ there exists $n_0(\varepsilon,\,\lambda) \in \mathbb{N}$ such that

$$F_{x_{n+p},x_n}(\varepsilon) > 1 - \lambda, \text{ for every } n \geq n_0(\varepsilon,\,\lambda) \text{ end every } p \in \mathbb{N}.$$

If a probabilistic metric space $(S,\, \mathcal{F})$ is such that every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in S converges in S, then $(S,\, \mathcal{F})$ is a complete space.

3 A fixed point theorem

In [4] the following theorem is proved:

Theorem 8 *Let T be a t-norm. Then (i) and (ii) hold, where:*

(i)Suppose that there exists a strictly increasing sequence $(b_n)_{n\in\mathbf{N}}$ from the interval $[0, 1)$ such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n, b_n) = b_n$. Then t-norm T is of H-type.

(ii) If t-norm T is continuous and of H-type, then there exists a sequence $(b_n)_{n\in\mathbb{N}}$ as in (i).

A theorem which we have proved is a generalization of a theorem given in [3].

Theorem 9 *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that t-norm T is continuous and of H-type, $f : S \to S$ a continuous mapping, $\Phi_n : S \to \mathbb{R}^+$ $(n \in \mathbb{N})$, and μ a mapping of $\mathbb{R}^+$ onto $\mathbb{R}^+$, such that μ is non-decreasing and*

$$\mu(a + b) \leq \mu(a) + \mu(b)$$

for every $a, b \in \mathbb{R}^+$. If for every $x \in S$, every $s > 0$ and every $n \in \mathbb{N}$

$$\mu(s) > \Phi_n(x) - \Phi_n(f(x)) \Rightarrow F_{x,f(x)}(s) > b_n$$

where $(b_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence from $(0, 1)$ such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n, b_n) = b_n$ for every $n \in \mathbb{N}$. Then there exists a a fixed point $x^* \in S$ of the mapping f and $x^* = \lim_{n\to\infty} f^n(x_0)$ for arbitrary $x_0 \in S$.

In the next theorem $(b_n)_{n\in\mathbf{N}}$ is monotone increasing sequence from $(0, 1)$ such that $\lim_{n\to\infty} b_n = 1$ but the members of sequence $(b_n)_{n\in\mathbb{N}}$ are not idempotent elements of the mapping T in a general case.

Theorem 10 *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that t-norm T is of H-type, $f : S \to S$ a continuous mapping, $\Phi_n : S \to \mathbb{R}^+$ $(n \in \mathbb{N})$, and μ a mapping of $\mathbb{R}^+$ onto $\mathbb{R}^+$, such that μ is non-decreasing and*

$$\mu(a + b) \leq \mu(a) + \mu(b)$$

for every $a, b \in \mathbb{R}^+$. If for every $x \in S$, every $s > 0$ and every $n \in \mathbb{N}$

$$\mu(s) > \Phi_n(x) - \Phi_n(f(x)) \Rightarrow F_{x,f(x)}(s) > b_n \tag{2}$$

then there exists an $x^* \in S$ such that $x^* = f(x^*)$ and $x^* = \lim_{n\to\infty} f^n(x_0)$, for arbitrary $x_0 \in S$.

Proof: We shall prove that (2) implies

$$\mu[d_n(x, f(x))] \leq \Phi_n(x) - \Phi_n(f(x)), \tag{3}$$

for every $n \in \mathbb{N}$ and every $x \in S$, where d_n is defined by

$$d_n(x,\, y) = \sup\{u \,|\, u \in \mathbb{R},\ F_{x,y}(u) \le b_n\}. \tag{4}$$

In order to prove (3) we shall prove the following implication:

$$s > \Phi_n(x) - \Phi_n(f(x)) \Rightarrow \mu[d_n(x,\, f(x))] \le s. \tag{5}$$

Let $s > \Phi_n(x) - \Phi_n(f(x))$. Since μ maps $\mathbb{R}^+$ onto $\mathbb{R}^+$, there exists $s_1 > 0$ such that

$$\mu(s_1) = s > \Phi_n(x) - \Phi_n(f(x)).$$

As from (2) it follows that $F_{x,fx}(s_1) > b_n$, we obtain that $d_n(x,\, f(x)) < s_1$ and therefore

$$\mu[d_n(x,\, f(x))] \le \mu(s_1) = s.$$

Therefore (5) holds.
Since t-norm T is of H-type it follows that for every $\alpha \in (0,\, 1)$, there exists $\beta \in (0,\, 1)$ such that

$$\underbrace{T(T(\ldots T}_{n-time}(1-\beta,\, 1-\beta), \ldots, 1-\beta) > 1-\alpha,$$

for every $n \in \mathbb{N}$. Let $1-\alpha = b_n$ where $n \in \mathbb{N}$. Then there exists $s(n) \in \mathbb{N}$ such that $1-\beta \le b_{s(n)}$, so

$$\underbrace{T(T(\ldots T}_{n-time}(b_{s(n)},\, b_{s(n)}), \ldots, b_{s(n)}) > b_n,$$ for every $n \in \mathbb{N}$. We shall prove that for every final set $\{v_1,\, v_2, \ldots v_m\} \subset S$, is

$$d_n(v_1,\, v_m) \le \sum_{i=1}^{m-1} d_{s(n)}(v_i,\, v_{i+1}). \tag{6}$$

Let $u_1,\, u_2, \ldots u_{m-1} \in \mathbb{R}$ be such that

$$\begin{aligned} d_{s(n)}(v_1,\, v_2) &< u_1 \\ d_{s(n)}(v_2,\, v_3) &< u_2 \\ &\cdots \\ d_{s(n)}(v_{m-1},\, v_m) &< u_{m-1}. \end{aligned}$$

Then from the definition (4) we have

$$
\begin{aligned}
F_{v_1,v_2}(u_1) &> b_{s(n)}\\
F_{v_2,v_3}(u_2) &> b_{s(n)}\\
&\dots\\
F_{v_{m-1},v_m}(u_{m-1}) &> b_{s(n)},
\end{aligned}
$$

i.e. $F_{v_1,v_m}(u_1+u_2+\dots u_m) \geq \mathbf{T}_{i=1}^{m} b_{s(n)} > b_n$, i.e. $d_n(v_1,\, v_m) \leq u_1+u_2+\dots+u_m$. So (6) is valid.

Let $x_0 \in S$ and $x_m = f^m(x_0)$ $(m \in \mathbb{N})$. Then for every $(m \in \mathbb{N})$

$$
\begin{aligned}
\mu[d_n(x_{m+1},\, x_m)] &= \mu[d_n(f(x_m),\, x_m)]\\
&\leq \Phi_n(x_m) - \Phi_n(f(x_m)),
\end{aligned}
$$

and so for every $k \in \mathbb{N}$

$$
\begin{aligned}
\sum_{i=0}^{k} \mu[d_n(x_{i+1},\, x_i)] &\leq \Phi_n(x_0) - \Phi_n(x_{k+1})\\
&\leq \Phi_n(x_0).
\end{aligned}
$$

Since μ is sub-additive it follows that

$$
\mu[\sum_{i=0}^{k} d_n(x_{i+1},\, x_i)] \leq \Phi_n(x_0). \tag{7}
$$

Relation (7)implies that

$$
\sum_{i=0}^{k} d_n(x_{i+1},\, x_i) \leq \sup\{u \mid u > 0,\, \mu(u) = \Phi_n(x_0)\} = M_n,
$$

and so the series

$$
\sum_{i=0}^{\infty} d_n(x_{i+1},\, x_i) \tag{8}
$$

is convergent.
Condition (8) is valid for every $n \in \mathbb{N}$ so it is satisfied for $s(n)$ and

$$
\begin{aligned}
d_n(x_m,\, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} d_{s(n)}(x_i,\, x_{i+1})\\
&\leq \sum_{i=m}^{\infty} d_{s(n)}(x_i,\, x_{i+1}).
\end{aligned}
$$

From the condition (6) it follows that $(x_n)_{n\in\mathbf{N}}$ is Cauchy sequence. If $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0)$, then from the continuity of f it follows that $x^* = f(x^*)$.

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