Max-Product Shepard Approximation Operators

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Abstract. In crisp approximation theory the operations that are used are only the usual sum and product of reals. We propose the following problem: are sum and product the only operations that can be used in approximation theory? As an answer to this problem we propose max-product Shepard Approximation operators and we prove that these operators have very similar properties to those provided by the crisp approximation theory. In this sense we obtain uniform approximation theorem of Weierstrass type, and Jackson-type error estimate in approximation by these operators.

1 Introduction

The main problem solved in crisp approximation theory is to approximate a

function $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a real interval, by some simpler function, e.g. (trigonometric) polynomial, rational function or wavelet. Crisp approximation theory provides many different approximation operators: Bernstein polynomials, Shepard-type rational approximation operators, trigonometric polynomials of Fejér type, wavelets, (see e.g. [3]) to mention only a few. These operators are using exclusively sum and product of reals as operations, and so, the linear algebra as underlying algebraic structure. Usually, the form of such an operator is

$$L(f, x) = \sum_{i=0}^{n} l_{n,i}(x) \cdot f(x_i),$$

where $x_i \in [a, b]$ are the knots, $i = 0, ..., n$, and $l_{n,i}(x)$ are functions having relatively simple expression (polynomials, trigonometric polynomials, rational functions, wavelets). The main theorems in crisp approximation theory are the Weierstrass-type uniform approximation theorems, which state that any continuous function can be approximated uniformly by operators of a given type, and error estimates which usually are given in terms of the modulus of continuity. Let us remark also that the approximation operators provided by crisp approximation theory are all linear.

Max t-norm compositions play a very important role in fuzzy logic and they were extensively studied, mainly in view of fuzzy relational equations (see e.g. [4]). Also, their applications to fuzzy control are well known (see [8]). The approximation capabilities of fuzzy systems (i.e. the capability of a fuzzy system to approximate some target function) are also well known ([6], [11], [5], [1], [7]).

These ideas lead us to propose the following question: Are sum and product the only operations that can be used in approximation theory? The answer is surely negative and in this sense we present max-product Shepard approximation operators, which use max instead of sum. For these operators we obtain Weierstrass-type uniform approximation theorem and for the approximation error we obtain Jackson-type error estimates in terms of the modulus of continuity. Since the operations used in the construction of these operators are max and product, the underlying algebraic structure is a max-product algebra and the mathematical analysis on these structures (i.e. the metric space structure) is usually called pseudo-analysis ([9]). Let us remark that these aproximation operators are nonlinear in contrast with the operators provided by crisp approximation theory.

After a preliminary section we introduce in Section 3 the max-product approximation operators and we study approximation properties of these operators. Some conclusions and further research topics conclude the paper.

2 Preliminaries

The purpose of this paper is to approximate a target function $f : X \to [0, \infty]$, where (X, d) is an arbitrary compact metric space, $[0, \infty]$ is endowed with max and product as algebraic operations and the usual topology induced by the Euclidean distance over the reals. So, the algebraic structure over $[0, \infty]$ is the max-product algebra. If we endow the max-product algebra with the topological structure induced by the Euclidean distance, we can use the tools of mathematical analysis. The mathematical analysis over this algebraic-topological structure is called pseudoanalysis (see [9]). The target function $f : X \to [0, \infty]$ is assumed to be continuous.

Usually, the error estimates in crisp approximation theory are provided in terms of the modulus of continuity. So, let us recall it's definition and main properties adapted to our case (for the general definition see [2]).

Definition 1 *Let (X, d) be a metric spaces and $([0, \infty], |\cdot|)$ the metric space of positive reals endowed with the usual Euclidean distance. Let $f : X \to [0, \infty]$ be* *a function. Then the function* $\omega(f,\cdot):[0,\infty)\to[0,\infty)$, *defined by*

$$\omega(f,\delta)=\bigvee\{|f(x)-f(y)|;x,y\in X,\ d(x,y)\leq\delta\}$$

is called the modulus of continuity of f.

Theorem 2 *The following properties hold true*
i) $|f(x)-f(y)|\leq\omega(f,d(x,y))$ *for any* $x,y\in X$*;*
ii) $\omega(f,\delta)$ *is nondecreasing in* δ*;*
iii) $\omega(f,0)=0$*;*
iv) $\omega(f,\delta_1+\delta_2)\leq\omega(f,\delta_1)+\omega(f,\delta_2)$ *for any* $\delta_1,\delta_2,\in[0,\infty)$*;*
v) $\omega(f,n\delta)\leq n\omega(f,\delta)$ *for any* $\delta\in[0,\infty)$ *and* $n\in\mathbb{N}$*;*
vi) $\omega(f,\lambda\delta)\leq(\lambda+1)\cdot\omega(f,\delta)$ *for any* $\delta,\lambda\in[0,\infty)$*;*
vii) If f *is continuous then* $\lim_{\delta\to0}\omega(f,\delta)=0$.

In order to study approximation properties of the operators defined later in this paper we need the following lemma.

Lemma 3 *For any functions* $A,B:\{0,...,n\}\to\mathbb{R}_+$ *we have*

$$\left|\bigvee_{i=0}^{n}A(i)-\bigvee_{i=0}^{n}B(i)\right|\leq\bigvee_{i=0}^{n}|A(i)-B(i)|,$$

for any $n\in\mathbb{N}^*$.

Proof. We observe that

$$\bigvee_{i=0}^{n}A(i)=\bigvee_{i=0}^{n}|B(i)+A(i)-B(i)|\leq\bigvee_{i=0}^{n}B(i)+\bigvee_{i=0}^{n}|A(i)-B(i)|,$$

inequality which, together wit the symmetric case, implies the statement of the lemma. ■

3 Max-product Shepard approximation operators

Let $f:X\to[0,\infty]$ be a continuous function. The Shepard-type max-product operator associated to f is defined by

$$Sh(f,n)(x)=Sh(x)=\bigvee_{i=0}^{n}\left(\frac{\frac{1}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}\cdot f(x_i)\right)=\frac{\bigvee_{i=0}^{n}\frac{f(x_i)}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}},\qquad(1)$$

where $\lambda \in \mathbb{N}^*$. It is easy to see that these are nonlinear, continuous operators. For simplicity of the notation we ommit the arguments f, n. In what follows we obtain the main results on these approximation operators.

The following Lemma is useful in obtaining the uniform approximation theorem of Weierstrass type.

Lemma 4 *For the approximation by Shepard-type max-product operator we have the following error bound*

$$\left|Sh\left(x\right)-f\left(x\right)\right| \leq \left(m\bigwedge_{i=0}^{n} d(x,x_i)+1\right)\cdot\omega\left(f,\frac{1}{m}\right), \tag{2}$$

for any $m \in \mathbb{N}$.

Proof. By Lemma 3 we have

$$\left|Sh\left(x\right)-f\left(x\right)\right| = \left|\frac{\bigvee_{i=0}^{n}\frac{f\left(x_i\right)}{d(x,x_i)^{\lambda}}-\bigvee_{i=0}^{n}\frac{f\left(x\right)}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}\right| \leq \frac{\bigvee_{i=0}^{n}\frac{\left|f\left(x\right)-f\left(x_i\right)\right|}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}.$$

By the properties of the modulus of oscillation of a function, we get for any $m \in \mathbb{N}$

$$\left|Sh\left(x\right)-f\left(x\right)\right| \leq \frac{\bigvee_{i=0}^{n}\frac{\omega\left(f,d(x,x_i)\right)}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}} \leq \frac{\bigvee_{i=0}^{n}\frac{\left(md(x,x_i)+1\right)\cdot\omega\left(f,\frac{1}{m}\right)}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}.$$

By direct computation we get

$$\left|Sh\left(x\right)-f\left(x\right)\right| \leq \frac{\bigvee_{i=0}^{n}\frac{\left(md(x,x_i)+1\right)}{d(x,x_i)^{\lambda}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}\cdot\omega\left(f,\frac{1}{m}\right) \leq$$

$$\leq \left(m\frac{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda-1}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}}+1\right)\cdot\omega\left(f,\frac{1}{m}\right).$$

It is easy to check that

$$\frac{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda-1}}}{\bigvee_{i=0}^{n}\frac{1}{d(x,x_i)^{\lambda}}} \leq \bigwedge_{i=0}^{n} d(x,x_i)$$

and we obtain

$$|Sh\,(x) - f\,(x)| \leq \left(m\bigwedge_{i=0}^{n} d(x,x_i) + 1 \right) \cdot \omega \left(f, \frac{1}{m} \right).$$

■

The following theorem is a uniform approximation theorem of Weierstrass type.

Theorem 5 *Any continuous function $f : X \to [0,\infty]$, can be uniformly approximated by Shepard-type max-product approximation operators, i.e. for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and a sequence of points x_i, $i = 0, ..., n$, such that $|Sh(f,n)(x) - f(x)| < \varepsilon$.*

Proof. Since X is a compact metric space, it is also totally bounded, i.e. for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and a finite covering of X by open balls B_i having radius ε and center x_i, $i = 0, ..., m$. If $\varepsilon = \frac{1}{m}$. then $\bigwedge_{i=0}^{n} d(x,x_i) < \frac{1}{m}$ and by the previous Lemma 4 we get

$$|Sh\,(x) - f\,(x)| \leq 2 \cdot \omega \left(f, \frac{1}{m} \right).$$

By Theorem 2 $\omega\left(f, \frac{1}{m}\right) \to 0$ for $m \to \infty$ and the proof is complete. ■

In what follows, we consider the case of equally spaced data in $[0,1]$ interval. In this case Jackson-type error estimate is obtained, that is the approximation error is proportional to $\omega\left(f, \frac{1}{n}\right)$ (see [3]). This result is important since it shows that by changing the operations we do not loose approximation properties, since in [10] the same order of the estimate is obtained for the classical Shepard approximation operators.

Theorem 6 *If $f : [0,1] \to [0,\infty]$ is continuous and $x_i = \frac{i}{n}$, $i = 0, ..., n$, then we have*

$$|Sh\,(x) - f\,(x)| \leq \frac{3}{2}\omega \left(f, \frac{1}{n} \right).$$

Proof. Since

$$\bigwedge_{i=0}^{n} \left| x - \frac{i}{n} \right| \leq \frac{i}{2n},$$

by taking $m = n$ in Lemma 4, we have

$$|Sh(x) - f(x)| \leq \frac{3}{2}\omega\left(f, \frac{1}{n}\right).$$

■

4 Concluding remarks and further research

The above obtained results show that sum and product are not the only operations that can be used in approximation theory. Indeed, by using max and product as operations, we defined a Shepard-type approximation operator. Moreover the Weierstrass-type approximation theorem and the Jackson-type error estimates obtained in this paper show us that we do not lose approximation properties. Also, since the operator is nonlinear it is possible that it provides better approximation for some function (it is well-known that using e.g. polynomial approximation for the solution of a nonlinear differential equation leads to loss of many properties).

Image processing uses as one of its usual tools approximation theory. So if we provide an approximation method then it is immediately interpreted as an image compression method. So we propose as a further research topic the efficient implementation of max-product approximation operators in image compression.

As further reserch topic we propose also the following question: Which are the best operations for approximation purposes for some given class of functions. As good candidates in this research we mention Frank t-norms. These t-norms have a Lipschitz-type property that can be helpful for approximation purposes.

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