Pseudo-Laplace type transform based on a pair of generated pseudo-operations with two parameters as pseudo-aggregation operator

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Abstract: Pseudo-Laplace transform is an important notion from pseudo-analysis framework that is often used in dealing with differential or integral equation. The $(\oplus, \odot)$*-Laplace transform presented here is a generalization of the pseudo-Laplace transform based on the special class of generalized pseudo-operations that need not be commutative nor associative. Generalization of exchange formula has been given. Also, the pseudo-Laplace type transforms in question has been used for construction of pseudo-aggregation operators.*

Keywords: Generated pseudo-operations with two parameters, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, pseudo-aggregation operators.

1 Introduction

Approach presented in this paper has been done in the pseudo-analysis' framework, where by pseudo-analysis a mathematical theory that is a generalization of the classical analysis has been considered. Pseudo-analysis has proved itself to be a useful tool for solving problems in different aspects of mathematics as well as in various practical problems ([7, 10, 13, 14]). It is based on a substitution of the field of real numbers with a semiring on the real interval, i.e., with a structure $([a,b], \oplus, \odot)$ where $[a,b]$ is closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and $\oplus$ and $\odot$ are pseudo-addition and pseudo-multiplication, respectively. Using this apparatus, over the years, some important notions that are analogous to their classical counterparts, i.e., notions as $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc., have been introduced ([7, 10, 13, 17, 18, 19]). Generalized pseudo-convolution, based on $\oplus$-measure and pseudo-integral, has taken an important role in theory of fuzzy numbers (operations with fuzzy numbers), as well as in optimization, information theory, system theory, etc. ([19]).

Also, pseudo-convolution and pseudo-Laplace transform have been applied to determining extreme values of utility functions ([5, 18]).

Of special interest is application of pseudo-analysis on nonlinear partial differential equations. By using the pseudo-linear superposition principle ([4, 8, 10, 13, 14, 15, 16, 17]) some new solutions of the considered nonlinear equation had been obtained. Also, the pseudo-analysis' approach had been successful in finding weak solution of Hamilton-Jacobi equation with non-smooth Hamiltonian ([10, 16, 18]). A step further in this direction had been presented in [22, 23] where generalized pseudo-operations were introduced. A special class of this operations that need not be commutative nor associative had been used to extend the pseudo-linear superposition principle on a generalized burger's type nonlinear partial differential equations [23]. Based on this special class of generalized pseudo-operations, corresponding measure, integral and convolution had been introduced in [20].

Another important problem is a problem of constructing aggregation operators by means of different types of pseudo-integrals. It is a well known fact that a large class of idempotent aggregation operators can be constructed and represented by different types of integrals. Some of the integrals that had been used for this constructions are Lebesgue integral, Choquet and Sugeno integral, monotone set functions-based integrals, Choquet-like integrals, (S, U)-integral etc. (see [1, 2, 3, 6, 11]).

This paper presents a generalization of the pseudo-Laplace transform based on a special class of generalized pseudo-operations, i.e., on a pair of generated pseudo-operations with two parameters. Also, the $\oplus$-integral as a core of pseudo-aggregation operator has been considered.

Preliminary notions as generalized pseudo-operations, $\oplus$-integral and corresponding pseudo-convolutions are given in Section 2. The third section contains definition of the $(\oplus, \odot)$-Laplace transform, where $\oplus$ and $\odot$ are generated pseudo-operations with two parameters. Generalization of the exchange formula that transforms convolution in to the product is given in Section 3. Aggregation type operator constructed by means of $(\oplus, \odot)$-Laplace transform is presented in the fourth section.

2 Preliminary notions

As already mentioned, through this paper the following special class of generalized pseudo-operations (see [22, 23]) will be considered.

Definition 1 *Let ε and γ be arbitrary but fixed positive real numbers and g a positive strictly monotone continuous function defined on $\mathbb{R}$ or $[0, \infty)$.* Generated pseudo-operations with two parameters $\oplus$ *and* $\odot$ *are*

$$x \oplus y = g^{-1}(\varepsilon g(x) + g(y)) \quad \textit{and} \quad x \odot y = g^{-1}(g(x)^{\gamma} g(y)).$$

Specially, for $\varepsilon = \gamma = 1$ operations from g-semiring are obtained ([9, 12, 13]).

Since operations $\oplus$ and $\odot$ need not be commutative nor associative operations, it is necessary to define pseudo-sum of n elements $\alpha_i \in [a, b]$, $i \in \{1, 2, \dots n\}$:

$$\bigoplus_{i=1}^{n} \alpha_i = (\dots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \dots) \oplus \alpha_n.$$

Neutral elements from the left for $\oplus$ and $\odot$ are $\mathbf{0} = g^{-1}(0)$ and $\mathbf{1} = g^{-1}(1)$, respectively, i.e., $\mathbf{0} \oplus x = x$ and $\mathbf{1} \odot x = x$.

Let $(a, b]$ be subinterval of the real line and let, for some $n \in \mathbb{N}$, $P_n = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ be its n-partition where $a = x_0 < x_1 < \dots < x_n = b$. Now, for ν being Lebesque measure, the $\oplus$-*measure* $\mu_{P_n} : P_n \to [0, \infty)$ has the following form:

$$\mu_{P_n}((x_i, x_{i+1}]) = g^{-1}\left(\frac{x_{i+1} - x_i}{\varepsilon^{n-i-1}}\right).$$

Some properties of this family of measures has been proved in [20]. Among them is the following pseudo-additive property

$$\mu_{P_{n-r+j}}\left(\bigcup_{i=j}^{r} A_i\right) = \bigoplus_{i=j}^{r} \mu_{P_n}(A_i),$$

where $1 \le j \le r \le n$, $P_n = \{A_i\}_{i=1}^{n} = \{(x_{i-1}, x_i]\}_{i=1}^{n}$ is a n-partition of interval $(a, b]$ and $P_{n-r+j} = \{B_s\}_{s=1}^{n-r+j}$ is new $(n - r + j)$-partition, such that $B_s = A_s$ while $s = 1, 2, \cdots, j - 1$, $B_j = \cup_{i=j}^{r} A_i$ and $B_s = A_{s+r-j}$ for $s = j + 1, \cdots, n - r + j$.

Let $\varphi : [a, b] \to [0, \infty)$ be a step function, i.e., φ is a measurable function that assumes only finitely many values $\{u_1, u_2, \dots, u_n\}$. Let us suppose that φ assumes value u_i for all $x \in (x_{i-1}, x_i]$, $i \in \{1, 2, \dots, n\}$, where $a = x_0 < x_1 < \dots < x_n = b$ is one n-partition of interval $(a, b]$. The $\oplus$-*integral of the step function* φ with respect to $\oplus$-measure μ_{P_n} is given by

$$\int_{[a,b]}^{(\oplus,\odot)} \varphi \, d\mu_{P_n} = \bigoplus_{i=1}^{n} u_i \odot \mu_{P_n}((x_{i-1}, x_i]). \tag{1}$$

Remark 2 Since the form of partition P_n from (1) follows directly from the form of step function φ, integral in (1) will be denoted with $\int_{[a,b]}^{(\oplus,\odot)} \varphi$, and, by means of the generating function g, it can be written as

$$\int_{[a,b]}^{(\oplus,\odot)} \varphi = g^{-1}\left(\sum_{i=1}^{n} (g(u_i))^{\gamma} (x_i - x_{i-1})\right).$$

With P'_n is denoted an $(n+1)$-partition of interval $(a,b]$ obtained from n-partition P_n in the following manner: we keep all the points from previous partition and add one more point and renumerate the points of the new partition in the increasing order. After s-repetition of the this procedure an $(n+s)$-partition $P_n^{(s)}$ is obtained (see [20]).

Let $f : [a,b] \to [0,\infty)$ be a continuous function. The $\oplus$-*integral* of function f is

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = \lim_{\substack{\mu_{P_n^{(s)}} \to 0 \\ (s \to +\infty)}} \left(\bigoplus_{i=0}^{n+s-1} \left(f(x_{i+1}) \odot \mu_{P_n^{(s)}}((x_i, x_{i+1}]) \right) \right), \tag{2}$$

if the limit exists.

Remark 3 Limit in (2) is considered with respect to the metric based on generated pseudo-operations with two parameters.

Since it has been proved in [20] that the $\oplus$-integral does not depend on the partition of the interval $[a,b]$ and that it can be represented in the following manner

$$\int_{[a,b]}^{(\oplus,\odot)} f d\mu_{P_n} = g^{-1}\left(\int_a^b g^{\gamma} \circ f(x) dx\right),$$

further on the $\oplus$-integral will be denoted by $\int_{[a,b]}^{(\oplus,\odot)} f$.

Corresponding *pseudo-convolution* of continuous functions $f, h : [0,\infty) \to [0,\infty)$ is

$$f \star h(x) = \int_{[0,x]}^{(\oplus,\odot)} ([f]_g(x-t) \odot h(t)), \tag{3}$$

where $[\cdot]_g$ is a transform given by $[f]_g(x) = g^{-1}\left(g^{1/\gamma}(f(x))\right)$.

3 The $(\oplus, \odot)$-Laplace transform

Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function g.

Definition 4 *The* $(\oplus,\odot)$**-Laplace transform** *of a continuous function* $f:[0,\infty)\to[0,\infty)$ *with respect to the* $\oplus$*-measure is*

$$\mathcal{L}^{\oplus}_{\odot}(f)(z)=\lim_{b\to\infty}\int_{[0,b]}^{(\oplus,\odot)}\Big(\left[g^{-1}\right]_g\left(e^{-xz}\right)\odot f(x)\Big), \tag{4}$$

if the limit exists.

Using connection between $\oplus$-integral and the Riemann integral, following form of $(\oplus,\odot)$-Laplace transform is obtained:

$$\mathcal{L}^{\oplus}_{\odot}(f)(z)=g^{-1}\left(\int_0^{\infty}e^{-xz\gamma}g\left(f(x)\right)^{\gamma}\,dx\right).$$

It can be proved that the pseudo-exchange formula for $(\oplus,\odot)$-Laplace transform given by Definition 2, i.e., formula that transforms $\oplus$-convolution in to pseudo-product, holds.

Theorem 5 *Let* $\oplus$ *and* $\odot$ *be generated pseudo-operations with two parameters given by generating function* g, $\mathcal{L}^{\oplus}_{\odot}$ *corresponding transform given by (4),* $\star$ *pseudo-convolution given by (3) and* $f_1,f_2:[0,\infty)\to[0,\infty)$ *continuous functions. Then, following holds*

$$\mathcal{L}^{\oplus}_{\odot}\left[f_1\star f_2\right]_g(z)=\left[\mathcal{L}^{\oplus}_{\odot}f_1\right]_g(z)\odot\mathcal{L}^{\oplus}_{\odot}f_2(z).$$

Proof follows from Definition 2 and properties of the classical Laplace transform.

Example 6 Let $\oplus$ and $\odot$ be generated pseudo-operations with two parameters given by generating function $g(x)=x^p,\ x\in[0,\infty)$ for some $p>0$. Under this assumption, corresponding $\mathcal{L}^{\oplus}_{\odot}$-transform of function $f:[0,\infty)\to[0,\infty)$ is

$$\mathcal{L}^{\oplus}_{\odot}(f)(z)=\left(\int_0^{\infty}e^{-xz\gamma}\left(f(x)\right)^{p\gamma}\,dx\right)^{1/p}.$$

It can be easily shown that exchange formula holds.

Remark 7 Specially, for $\varepsilon=\gamma=1$ pseudo-Laplace transform from [18] can be obtained. In this case, the pseudo-exchange formula in cooperation with the inverse pseudo-Laplace transform had been applied to determining extreme values of utility functions ([5, 18]).

Remark 8 Another generalization of Laplace type transform of a measurable function $f:[0,\infty)\to[0,1]$ known as the (S,T)-Laplace transform, where $([0,1],S,T)$ is the conditionally distributive semiring, can be found in [5].

Remark 9 In [21] one more direction for generalization of the pseudo-Laplace type transform has been presented. This generalization is done on the domain of functions that pseudo-Laplace type transform has been applied to. In this case $([a,b],\oplus,\odot)$ is a semiring from the first or second class (see [7, 10, 13, 16, 17, 18, 19]), L is a binary operation on $[0,+\infty)$ which is non-decreasing in both coordinate, continuous on $[0,+\infty)^2$, commutative, associative, has 0 as identity, fulfills cancellation law and is given by multiplicative generator $l:[0,\infty]\to[0,1]$ as $L(x,y)=l^{-1}\left(l(x)l(y)\right)$, and $\diamondsuit$ is another binary operation $[0,\infty)$ distributive with respect to L. For $\oplus=\max$ and $\odot$ being an Archimedean t-norm T given by continuous and increasing generating function $\theta:[0,1]\to[0,1]$ (see [5]), generalized $(\max,T)$ -Laplace transform from [21] is mapping $\mathcal{L}^{\max}_{T,L}$ defined for all $F:[0,\infty)\to[0,1]$ as

$$\mathcal{L}^{\max}_{T,L}F(z)=\theta^{(-1)}\left(\sup_{x\geq 0} l(x\diamondsuit z)\theta\left(F(x)\right)\right),\quad z\geq 0.$$

If $\oplus$ and $\odot$ are strict pseudo-operations given by generating function g (semiring of the second class, see [16, 17]), the generalized $(\oplus,\odot)$ -Laplace transform from [21] is mapping $\mathcal{L}^{\oplus}_{\odot,L}$ defined for $F:[0,\infty)\to[a,b]$ as

$$\mathcal{L}^{\oplus}_{\odot,L}F(z)=g^{-1}\left(\int_{[0,\infty)} l(x\diamondsuit z)\theta\left(F(x)\right)\right)dx.$$

4 Pseudo-aggregation operators based on the $(\oplus,\odot)$-Laplace transforms

By an aggregation operator ([2]) is usually considered a function $A:\cup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$ such that

(i) $A(u_1,\ldots,u_n)\leq A(v_1,\ldots,v_2)$ when $u_i\leq v_i$ for all $i\in\{1,\ldots,n\}$,

(ii) $A(u)=u$ for all $u\in[0,1]$,

(iii) $A(1,\ldots,1)=1$ and $A(0,\ldots,0)=0$.

A large class of aggregation operators had been constructed by different types of integrals ([1, 2, 6]). Now, a method similar to the construction of (S,U)-integral-based aggregation operators ([6]) can be applied to construction of following $\oplus$-integral-based aggregation type operator.

Pseudo-aggregation operator $\widetilde{A}:\cup_{n\in\mathbb{N}}[0,\infty)^n\to[0,\infty)$ based on $\oplus$-integral is

$$\widetilde{A}(u_1,\ldots,u_n)=\int_{[0,1]}^{(\oplus,\odot)}\varphi, \tag{5}$$

where $\varphi : (0,1] \to [0,\infty)$ is a step function given by $\varphi(x) = u_i,\ x_{i-1} < x \le x_i,$ $i \in \{1,\dots,n\},$ for some n-partition $0 = x_0 < x_1 \dots < x_n = 1$ (see [2, 6]).

For operators given by (5) following hold:

(i) $A(u_1,\dots,u_n) \le A(v_1,\dots,v_2)$ when $u_i \le v_i$ for all $i \in \{1,\dots,n\}$,

(ii) $A(u) = u \odot \mathbf{1}$ for all $u \in [0,1]$,

(iii) $A(\mathbf{1},\dots,\mathbf{1}) = \mathbf{1}$ and $A(\mathbf{0},\dots,\mathbf{0}) = \mathbf{0}$,

where $\mathbf{0}$ and $\mathbf{1}$ neutral elements for pseudo-addition $\oplus$ and pseudo-multiplication $\odot$, respectively. Also, for pseudo-aggregation operators given by (5), some type of are pseudo-linearity and pseudo-additivety (with respect to pseudo-operations $\oplus$ and $\odot$ and transform $[\cdot]_g$) can be proved.

Example 10 Let us consider generating function $g : [0,\infty) \to [0,\infty)$ such that $g(x) = x^p$ for some $p > 0$. Corresponding pseudo-operations are $x \oplus y = (\varepsilon x^p + y^p)^{1/p}$ and $x \odot y = x^\gamma y$. If the length of each interval $(x_{i-1}, x_i]$ associated to the input value u_i is denoted with $l_i,\ i \in \{1,\dots,n\},$ operator $\widetilde{A}$ of n inputs $u_1, u_2, \dots, u_n$ has the following form

$$\widetilde{A}(u_1,\dots,u_n) = \left(\sum_{i=1}^{n} l_i u_i^{\gamma p}\right)^{1/p},$$

where $\sum_{i=1}^{n} l_i = 1$. If all the intervals $(x_{i-1}, x_i]$ are of the equal length, operator $\widetilde{A}$ is

$$\widetilde{A}(u_1,\dots,u_n) = \left(\frac{1}{n}\sum_{i=1}^{n} u_i^{\gamma p}\right)^{1/p}.$$

The question is whether operators of aggregation type can be induced by $(\oplus,\odot)$-Laplace transforms.

Let $u_1, u_2, \dots, u_n$ be n input values from $[0,\infty)$. For each n input values and each n-partition where $0 = x_0 < x_1 < \dots < x_n = 1$ of interval $(0,1]$ is possible to form a step function $\varphi : (0,\infty) \to [0,\infty)$ as

$$\varphi(x) = \begin{cases} u_i, & \text{for } x \in (x_{i-1}, x_i], \\ g^{-1}(0), & \text{for } x > 1, \end{cases} \tag{6}$$

where g is a generating function for pseudo-operations $\oplus$ and $\odot$ given by Definition 1.

Aggregation type operator of n input values will be induced by application of $(\oplus,\odot)$-Laplace transform to a step function associated to this n input values.

Definition 11 Pseudo-aggregation operator $\widetilde{A_L} : \cup_{n\in\mathbb{N}}[0,\infty)^n \to [0,\infty)$ based on $(\oplus,\odot)$-Laplace transform *is*

$$\widetilde{A_L}(u_1,\ldots,u_n) = \mathcal{L}^{\oplus}_{\odot}(\varphi)(z), \tag{7}$$

where φ is a step function for input values $u_1,u_2,\ldots,u_n$ given by (6) and z is some real positive parameter.

Since $(\oplus,\odot)$-Laplace transform is based on non-associative and non-commutative pseudo-operations, the impact of some input value to the result can be determined by its index and by length of associated subinterval of the unite interval.

It can be easily shown that pseudo-aggregation operator $\widetilde{A_L}$ with parameter z has the following form

$$\widetilde{A_L}(u_1,\ldots,u_n) = \bigoplus_{i=1}^{n} u_i \odot \omega_{(x_{i-1},x_i],z},$$

where $(x_{i-1},x_i]$ is subinterval of the unite interval is associated to input value u_i and

$$\omega_{(x_{i-1},x_i],z} = g^{-1}\left(\frac{e^{-z\gamma x_{i-1}} - e^{-z\gamma x_i}}{\varepsilon^{n-i}\gamma z}\right).$$

Basic properties of pseudo-aggregation operator $\widetilde{A_L}$ with parameter z are

(i) $\widetilde{A_L}(u_1,\ldots,u_n) \leq \widetilde{A_L}(v_1,\ldots,v_2)$ when $u_i \leq v_i$ for all $i \in \{1,\ldots,n\}$,

(ii) $\widetilde{A_L}(u) = u \odot \omega_{(0,1],z}$ for all $u \in [0,1]$,

(iii) $\widetilde{A_L}(\mathbf{1},\ldots,\mathbf{1}) = \mathbf{1} \odot \omega_{(0,1],z} = \omega_{(0,1],z}$ and $\widetilde{A_L}(\mathbf{0},\ldots,\mathbf{0}) = \mathbf{0} \odot \omega_{(0,1],z} = \mathbf{0}$,

where $\omega_{(0,1],z} = g^{-1}\left((1 - e^{-z\gamma})/\gamma z\right)$.

Conclusion

The main aim of this paper has been to present further possible steps in the generalization, based on the pseudo-analysis' apparatus, of well known notions as Laplace transform and aggregation operators that could broaden the area of applications. Some further research of this problem should concerne properties of $(\oplus,\odot)$-Laplace transform and pseudo-aggregation operators and possible applications.

Acknowledgement. This paper has been supported by the project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" by Academy of Sciences and Arts of Vojvodina supported by Vojvodina Provincial Secretariat for Science and Technological Development.

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