Newton's Method with Accelerated Convergence Modified by an Aggregation Operator

Tibor Lukić, Nebojša M. Ralević

Faculty of Engineering, University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad, Serbia and Montenegro
e-mail: tibor@uns.ns.ac.yu, nralevic@uns.ns.ac.yu

Abstract: In this paper a new modification of Newton's method based on aggregation operator-geometric mean, for solving nonlinear equations has proposed. The convergence properties of proposed method have been discussed and has shown that the order of convergence is three. Theoretically result has been verified on relevant numerical problems and comparison of the behavior of the proposed method and some existing ones are given.

Key words and phrases: Newton's method, Aggregation operators, Iterativ methods, Order of convergence, Geometric mean, Function evaluations.

1 Introduction

We consider the problem of numerical determine a real root α of nonlinear equation

$$f(x) = 0, \quad f : D \subseteq R \to R\,. \tag{1}$$

The best know numerical method for solving equation (1) is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{2}$$

where x_0 is an initial approximation sufficiently close to α. The convergence order of the classical Newton's method is quadratically for simple roots and linearly for multiple roots. In literature [2, 3, 10, 11] are given some variant of Newton's methods, which have a target to improve the rate of convergence or to make smaller the number of functional evaluations. The methods developed by Fernando et al. [10] and Özban [2] suggested the new method proposed in this work.

Definition 1 *(See [9]) If the sequence $\{x_n\}$ tends to a limit α in such a way that*

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = C$$

for some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is know as the asymptotic error constant.

If $p = 1$, $p = 2$ or $p = 3$, the convergence i said to be *linearly*, *quadratically* or *cubic*, respectively.
Let $e_n = x_n - \alpha$ be the error in the nth iterate of the method which produces the sequence $\{x_n\}$. Then, the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}) = O(e_n^p)$$

is called the *error equation*. The value of p is called the order of convergence of this method.

Definition 2 *(See [10]) Let α be a root of the function f and suppose that x_{n+1}, x_n and x_{n-1} are three consecutive iterations closer to the root α. Then the computational order of convergence ρ can be approximated using the formula*

$$\rho \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

In this work, as in scientific papers [2] and [10], Newton's method is modificated by aggregation operators like: geometric, arithmetic and harmonic means are.

Aggregation operators like: triangular norms and conorms, uninorms, copulas, weighted arithmetic means, ordered weighted arithmetic means and compensated operators are, make a special class of aggregation operators. All these operators are detailed considered in different scientific papers and monographs, for example monograph of E. P. Klement, R. Mesiar and E. Pap is dedicated to triangular norms [6], the ordered weighted averaging operators are considered in edition of R. R. Yager i J. Kacprzyk [7], while copulas are presented in monograph of R. B. Nelsen [8]. Many results connected with aggregation operations can be found in edition of T. Calva, G. Mayor and R. Mesiar [1].

We going to present a definition, some examples and properties of aggregation operators.

Definition 3 *An* **aggregation operator** *is a function $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ such that*

i) $A(x_1, ..., x_n) \leq A(y_1, ..., y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, ..., n\}$.

ii) $A(x) = x$ for all $x \in [0,1]$.

iii) $A(0,...,0) = 0$ *and* $A(1,...,1) = 1$.

Operators Π, as the operator of product, arithmetic mean M, Min, Max and operator A_c are all aggregation operators.

$$\Pi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i\,, \qquad M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i\,,$$

$$Min(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)\,, \qquad Max(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)\,,$$

$$A_c(x_1, \ldots, x_n) = \max(\,0,\, \min(\,1,\, c + \sum_{i=1}^{n}(x_i - c)))$$

where the operator $A_c : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is defined for all $c \in (0,1)$.

The weakest aggregation operator, in designation A_w, and the strongest aggregation operator A_s, are given by following:

$$\begin{array}{lll} (\forall\ n \geq 2) & (x_1, \ldots, x_n) \neq (1, \ldots, 1)\ :\ A_w(x_1, \ldots, x_n) = 0 \\ (\forall\ n \geq 2) & (x_1, \ldots, x_n) \neq (0, \ldots, 0)\ :\ A_s(x_1, \ldots, x_n) = 1. \end{array}$$

Aggregation operators between each other can compare like functions with n-variables. For any aggregation operator A is satisfied:

$$A_w \leq A \leq A_s.$$

Also, the following is satisfied:

$$A_w \leq \Pi \leq Min \leq M \leq Max \leq A_s.$$

Example 1 *Aggregation operator:* $W_\triangle : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ *defined by*

$$W_\triangle(\,x_1\,,\,x_2,\,\ldots,\,x_n) = \sum_{i=1}^{n} w_{in}x_i\,,$$

is the so called weighted arithmetic operator associated with weighted triangle $\triangle$.

Weighted arithmetic means are continuous, idempotent, linear, additive and self-dual aggregation operators.

Example 2 *Let* $f\ :\ [0,1] \to [-\infty, +\infty]$ *continuous and strictly monotone function. The aggregation operator* $M_f\ :\ \bigcup\limits_{n\in\mathbb{N}}[0,1]^n \to [0,1]$*, which is given by*

$$M_f(\,x_1,\,x_2,\,\ldots,\,x_n) = f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(\,x_i)\Big)\,,$$

is called **quasi-arithmetic mean**.

The class of quasi-arithmetic means, root-exponentials operators $M_p : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, $p \in (-\infty, 0) \cup (0, +\infty)$ is obtained by applying the function $f_p : [0,1] \to [-\infty, +\infty]$, $f_p(x) = x^p$ such as:

$$M_p(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}.$$

Marginal members of these classes are $M_0 = G = M_{\log x}$, which is the geometric mean, while $M_\infty = Max$ and $M_{-\infty} = Min$ which are not in class of quasi-arithmetic means.

2 Description of the method

2.1 Arithmetic and Harmonic mean Newton's Methods

Let α is a simple root of nonlinear equation $f(x) = 0$, where f is a sufficiently differentiable function. It is clear, from Newton's theorem that

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(z)\, dz\,. \tag{3}$$

If we approximate the definite integral in (3) with rectangle $(x - x_n)f'(x_n)$, and take $x = \alpha$, we have that

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n),$$

and if α declare by the new approximation x_{n+1}, we obtain the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Leading by such admission Fernando et al.[10] approximate the definite integral in (3) with trapezoid rule

$$\int_{x_n}^{x} f'(z)\, dz \approx \frac{1}{2}(x - x_n)[f'(x_n) + f'(x)],$$

and by such a way arrive the following method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(v_{n+1})}, \text{ where } v_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0, 1, \ldots, \tag{4}$$

which is, in contrast to Newton's method (2), instead of $f'(x_n)$ using arithmetic mean of $f'(x_n)$ and $f'(v_{n+1})$. Therefore, it call *arithmetic mean Newton's method* (AM). If we use harmonic mean instead of the arithmetic mean in (4), we obtain

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(v_{n+1}))}{2f'(x_n)f'(v_{n+1})}, \quad n = 0, 1, \ldots$$

which call *harmonic mean Newton's method* (HM) proposed by Özban [2].

2.2 New Modification of Newton's Method

If we use the geometric mean instead of arithmetic mean in (4), we get the new scheme

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_n)f'(v_{n+1})}}, \\ \text{where } \ v_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots \end{aligned} \tag{5}$$

which we call *geometric mean Newton's method* (GM).

3 Analysis of convergence

Theorem 1 *Let $f : D \subseteq R \to R$ for an open interval D. Assume that f is sufficiently differentiable function in the interval D and f has a simple root in $\alpha \in D$. If x_0 is sufficiently close to α, then the new method defined by (5) converges cubically and satisfies the following error equation:*

$$e_{n+1} = \left(c_2^2 - \frac{1}{2}c_2 - \frac{3}{2}c_3\right) e_n^3 + O(e_n^4), \tag{6}$$

where $e_n = x_n - \alpha$ and constants $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$ for $j = 1, 2, 3, \ldots$.

PROOF. Let α be a simple root of equation $f(x) = 0$ (i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$). Since the function f' is continuously in open interval D we choice the initial approximation x_0 sufficiently close to $\alpha \in D$ such that $sign(f'(x_0)) = sign(f'(\alpha))$.
By Taylor expansion of $f(x_n)$ about α we get

$$\begin{aligned} f(x_n) &= f(\alpha) + f'(\alpha)e_n + \tfrac{1}{2!}f''(\alpha)e_n^2 + \tfrac{1}{3!}f^{(3)}(\alpha)e_n^3 + O(e_n^4) \\ &= f'(\alpha)[e_n + c_2e_n^2 + 3c_3e_n^3 + O(e_n^4)], \end{aligned} \tag{7}$$

where $e_n = x_n - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$. Similarly, we obtain

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \ldots]. \tag{8}$$

Dividing (7) by (8), we get

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= [e_n + c_2e_n^2 + 3c_3e_n^3 + O(e_n^4)][1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \ldots]^{-1} \\ &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \end{aligned} \tag{9}$$

and

$$\begin{aligned} v_{n+1} &= x_n - \tfrac{f(x_n)}{f'(x_n)} \\ &= \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4). \end{aligned} \tag{10}$$

By (10) and expanding $f'(v_{n+1})$ about α we obtain

$$f'(v_{n+1}) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{11}$$

Multiplication (8) by (11), we get

$$f'(x_{n+1})f'(v_{n+1}) = f'^2(\alpha)[1+2c_2e_n+(2c_2^2+3c_3)e_n^2+4(c_2c_3+c_4)e_n^3+O(e_n^4)]\,,$$

and

$$\begin{aligned} & sign(f'(x_0))\sqrt{f'(x_{n+1})f'(v_{n+1})} \\ &= f'(\alpha)[1 + 2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O(e_n^4)]^{\frac{1}{2}} \\ &= f'(\alpha)[1 + \tfrac{1}{2}(2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O(e_n^4)) \\ & -\tfrac{1}{8}(2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O(e_n^4))^2 + O(e_n^3)] \\ &= f'(\alpha)[1 + c_2e_n + (c_2^2 - \tfrac{1}{2}c_2 + \tfrac{3}{2}c_3)e_n^2 + O(e_n^3)]. \end{aligned} \tag{12}$$

Hence, from equations (7) and (12), we have that

$$\begin{aligned} & \frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_{n+1})f'(v_{n+1})}} \\ &= [e_n + c_2e_n^2 + 3c_3e_n^3 + O(e_n^4)][1 + c_2e_n + (c_2^2 - \tfrac{1}{2}c_2 + \tfrac{3}{2}c_3)e_n^2 + O(e_n^3)]^{-1} \\ &= e_n + (\tfrac{1}{2}c_2 + \tfrac{3}{2}c_3 - c_2^2)e_n^3 + O(e_n^4). \end{aligned}$$

Replacing this in (5) we obtain

$$x_{n+1} = x_n - (e_n + (\frac{1}{2}c_2 + \frac{3}{2}c_3 - c_2^2)e_n^3 + O(e_n^4))$$

or

$$e_{n+1} = (c_2^2 - \frac{1}{2}c_2 - \frac{3}{2}c_3)e_n^3 + O(e_n^4)\,,$$

which shows the third-order convergence of the geometric mean Newton's method.

4 Numerical results and conclusions

Table 1.

Function	x_0	i				COC				NOFE			
		NM	AM	HM	GM	NM	AM	HM	GM	NM	AM	HM	GM
(a)	0.5	7	4	4	4	2.00	3.00	3.00	3.00	14	12	12	12
	1	5	3	3	3	1.98	3.00	2.90	3.00	10	9	9	9
	2	5	3	3	3	2.00	3.00	3.00	2.89	10	9	9	9
(b)	-1	6	3	3	4	2.00	ND	3.00	3.00	12	9	9	12
	-3	6	3	3	4	1.97	ND	3.00	3.00	12	9	9	12
(c)	-2	8	6	5	5	2.00	3.00	3.00	2.97	16	18	15	15
	-3	14	9	8	9	2.00	2.94	3.00	3.00	28	27	24	27
(d)	0	9	15	5	2	2.00	3.00	3.00	ND	18	45	15	6
	1.5	7	5	4	4	1.98	3.00	3.00	3.00	14	15	12	12
	2.5	6	4	3	4	2.00	2.90	3.00	2.97	12	12	9	12
	3.5	7	5	4	4	2.00	3.00	3.00	3.00	14	15	12	12
(e)	1.5	15	467	7	12	1.96	ND	3.00	3.00	30	1401	21	36
	2.5	7	5	4	5	1.98	ND	3.00	2.98	14	15	12	15
	3.5	10	7	6	6	2.00	2.98	3.00	3.00	20	21	18	18
(f)	1.4	78	51	41	46	1.00	1.00	1.00	1.00	156	153	123	138
	-3	113	75	60	67	1.00	1.00	1.00	1.00	226	225	180	201

NM - Newton's method
AM - Arithmetic mean N. method
HM - Harmonic mean N. method
GM - Geometric mean N. method
COC - Computational order of convergence
NOFE - Number of functional evaluations
i - Number of iterations
ND - Not defined

In Table 1. we show the computational results of some relevant numerical test to compare the efficiencies of the methods. The used stopping criterion is $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$.

Test functions (See [5, 10])

(a) $x^3 + 4x^2 - 10,$ $\alpha = 1.365230013414097,$
(b) $\sin^2 x - x^2 + 1,$ $\alpha = -1.404491648215341,$
(c) $xe^{x^2} - \sin^2 x + 3\cos x + 5,$ $\alpha = -1.207647827130919,$
(d) $(x-1)^3 - 1,$ $\alpha = 2,$
(e) $(x-1)^6 - 1,$ $\alpha = 2,$
(f) $(x-2)^3(x+2)^4,$ $\alpha = 2 \ \vee \ \alpha = -2.$

All numerical tests agree with the theoretically result of this paper. The most important characteristics of geometric mean Newton's method (GM) are:

(1) third order of convergence (for simple roots),
(2) does not require the computation of second or higher order derivatives,
(3) by the numerical results (Table 1.) it is evident that the total number of functional evaluation required is less than of Newton's method.

It is interesting to consider the behavior of tested methods for multiple roots.

The test function (f) has a multiple roots and the COC is linear. This is in accordance with the theoretically properties of Newton's method for multiple roots (see [4]). Further investigations will be related to other aggregation operator Newton's methods.

References

[1] T. Calvo, G. Mayor and R. Mesiar editors: *Aggregation Operators, New Trends and Applications,* Physica-Verlag, Heidelberg, 2002.

[2] A. Y. Özban, *Some new variants of Newton's method*, Appl. Math. Lett. 17, 677—682, 2004.

[3] J. Gerlach, *Accelerated convergence in Newton's method*, SIAM Review 36, 272—276, 1994.

[4] J. M. Ortega and W. C. Reinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[5] M. Igarashi, *A termination criterion for iterative methods used to find the zeros of polynomials*, Math. Comp. 42, 165—171, 1984.

[6] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[7] R.R. Yager i J. Kacprzyk: *The Ordered Weighted Averaging Operators, Theory and applications,* Kluwer Academic Publishers, Dordrecht, London, 1997.

[8] R. B. Nelsen: *An Introduction to Copulas,* Lecture Notes in Statistic 139, Springer, 1999.

[9] R. Wait , *The numerical Solution of Algebraic Equations*, John Wiley & Sons, 1979.

[10] S. Weerakoon and T. G. I. Fernando, *A variant of Newton's method with accelarated third-order convergence*, Appl. Math. Lett. 13, 87—93, 2000.

[11] W. F. Ford and J. A. Pennline, *Accelerated convergence in Newton's method*, SIAM Review 38, 658—659, 1996.