Difference representations of the Choquet integral with respect to a signed fuzzy measure

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Abstract: This paper provides a discussion on difference representations of the asymmetric Choquet integral with respect to a signed fuzzy measure with bounded chain variation. There are given difference representations of the Choquet integral with respect to a signed fuzzy measure based on its representation as difference of two fuzzy measures.

Key words and phrases: signed fuzzy measures, chain variation, Choquet integral.

1 Introduction

A general monotone (non-decreasing) non-negative set function, vanishing at the empty set, is called by various names, such as capacity, cooperative game, non-additive measure, fuzzy measure. In this paper we call them *fuzzy measures*. Generalized fuzzy measure, *a signed fuzzy measure* is revised monotone set function, vanishing at the empty set, and it can take negative values.

One of the most used integral based on a fuzzy measure m is the Choquet integral [4, 7, 10, 11]. The Choquet integral is often used in economics, pattern recognition and decision analysis as nonlinear aggregation tool [13]. Two crucial properties of the Choquet integral, defined for non-negative measurable functions, are monotonicity and comonotonic additivity, see [2, 3, 4, 10]. There exist two extensions of Choquet integral to the class of all measurable functions, *the symmetric* Choquet integral, introduced by Šipoš and *the asymmetric* Choquet integral, see [2, 10]. The second one is defined with respect to a real-valued set function m, not necessary monotone.

For the main field of application of Choquet integral, decision under uncertainty, an universal set X is a space, its elements are state of nature and functions from X to $\mathbb{R}$ are prospects. The preference relation $\preceq$ is defined on the set of prospects and we say that a utility functional L represents a preference relation if and only if $L(f) \leq L(g)$ for all pairs of prospects f, g such that $f \preceq g$. Schmeidler [15] showed that preference can be represented by Choquet integral model, so called Choquet expected utility model (cumulative utility). Choquet expected utility model is not an appropriate tool when the gain and loss must be considered in the same time. Cumulative Prospect Theory (CPT), introduced by Tversky and Kahneman [14], combines cumulative utility and a generalization of expected utility, so called sign dependent expected utility. CPT holds if the preference can be represented by the difference of two Choquet integrals, i.e.,

$$L(f) = C_{m^+}(f^+) - C_{m^-}(f^-), \tag{1}$$

where m^+ and m^- are two fuzzy measures, $f^+ = f \vee 0$ is the gain part of prospect f, and $-f^-$ is it's loss part, $f^- = (-f) \vee 0$. An corresponding difference formula to (1) but with Sugeno integrals was proved in [12].

Motivated by (1) the aim of this paper is to present some difference representations of asymmetric Choquet integral w.r.t a signed fuzzy measures. The paper is organized as follows. In the next section the short overview of basic notions and definitions has been given, and the difference formula (2) is presented. In Section 3. we introduce a chain variation of set functions and the space BV, the family of set functions, vanishing at the empty set, with bounded chain variation. In this section we consider a difference representation (3) of Choquet integral w.r.t a signed fuzzy measure m with bounded chain variation. In Section 4. an interpreter and a frame for representation of the signed fuzzy measures have been defined. We shall prove that for every signed fuzzy measure $m \in BV$ there exists a representation of m. Applying this result, we present another difference representation (4) of Choquet integral w.r.t m.

2 Preliminaries

Let X be an universal set. Let $\mathcal{A}$ be a σ-algebra of subsets of X. $(X, \mathcal{A})$ is called a measurable space [10]. A set function μ, $\mu : \mathcal{A} \to [-\infty, \infty]$ is called a *signed measure*, if for each sequence $E_1, E_2, \ldots$ of mutually disjoint sets from $\mathcal{A}$ the series $\sum_{i=1}^{\infty} \mu(E_i)$ is defined and the equality $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ holds, and μ assumes at most one of the values ∞ and $-\infty$.

A *fuzzy measure* m is a non-negative real-valued set function defined on a σ-algebra $\mathcal{A}$ with the following properties:

(FM1) $m(\emptyset) = 0$,

(FM2) $E \subset F \quad \to \quad m(E) \leq m(F)$, for all $E, F \in \mathcal{A}$

(FM3) $E_n \in \mathcal{A}\ ,\ E_n \nearrow E \ \to\ m(E_n) \nearrow m(E)$,

(FM4) $E_n \in \mathcal{A}\ ,\ E_n \searrow E$ and there exists n_0 such that $m(E_{n_0}) < \infty \ \to$ $m(E_n) \searrow m(E)$.

The condition (FM3) means that m is a fuzzy measure continuous from below and (FM4) means that m is continuous from above. In the sequel it has been assumed that m is generalized fuzzy measure, i.e., it satisfied (FM1), (FM2).

A set function m, $m : \mathcal{A} \to [-\infty, \infty]$ is called a *signed fuzzy measure* if m satisfies
(SFM1) $m(\emptyset) = 0$,
(SFM2) If $E, F \in \mathcal{A}$, $E \cap F = \emptyset$, then
a) $m(E) \geq 0$, $m(F) \geq 0$, $m(E) \vee m(F) > 0 \Rightarrow m(E \cup F) \geq m(E) \vee m(F)$;
b) $m(E) \leq 0$, $m(F) \leq 0$, $m(E) \wedge m(F) < 0 \Rightarrow m(E \cup F) \leq m(E) \wedge m(F)$;
c) $m(E) > 0$, $m(F) < 0$, $\Rightarrow m(F) \leq m(E \cup F) \leq m(E)$;
(SFM3) continuity from below
(SFM4) continuity from above
The condition (SFM2) of m is called revised monotonicity. In the sequel we assume that m is generalized signed fuzzy measure, i.e., it satisfied (SFM1), (SFM2).

The conjugate set function $\bar{m}$ of real-valued set function $m : \mathcal{A} \to \mathbb{R}$ is defined by $\bar{m}(E) = m(X) - m(\bar{E})$, where $\bar{E}$ denotes the complement set of E, $\bar{E} = X \setminus E$. Obviously, if m is a fuzzy measure, $\bar{m}$ is a fuzzy measure, too.

Let $\mathcal{M}$ be the class of all non-negative measurable functions f on X and let $\overline{\mathcal{M}}$ denotes the class of all measurable functions on X. We introduce the Choquet integral with respect to a fuzzy measure (respectively a signed fuzzy measure) $m : \mathcal{A} \to [0, \infty]$ (respectively $\mathbb{R}$) of a measurable function $f : X \to [0, \infty]$ (respectively $[-\infty, \infty]$).

Definition 1 *([2, 10]) Let $(X, \mathcal{A})$ be a measurable space.*

i) The Choquet integral *w.r.t a fuzzy measure $m : \mathcal{A} \to [0, \infty]$ is functional $C_m : \mathcal{M} \to [0, \infty]$ defined by*

$$C_m(f) = \int_0^\infty m(\{x | f(x) \geq t\}) \, dt$$

ii) The asymmetric Choquet integral *w.r.t a set function $m : \mathcal{A} \to \mathbb{R}$, is functional $C_m : \overline{\mathcal{M}} \to [-\infty, \infty]$ defined by*

$$C_m(f) = \int_{-\infty}^0 (m(\{x | f(x) \geq t\}) - m(X)) \, dt + \int_0^\infty m(\{x | f(x) \geq t\}) \, dt$$

if both of the above Lebesgue integrals exist. When the expression $\infty - \infty$ is occurred, the integral is not defined.

The asymmetric Choquet integral can be expressed in the terms of the Choquet integrals of non-negative functions f^+ and f^-, the positive and negative parts of the function f, i.e.

$$C_m(f) = C_m(f^+) - C_{\bar{m}}(f^-), \tag{2}$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$, and $\bar{m}$ is the conjugate set function of m.

3 Signed fuzzy measures with bounded chain variation

The chain variation of real-valued set functions, vanishing at the empty set, and the space BV will be introduced, see [1, 10].

Definition 2 *The chain variation of a real-valued set function m, $m(\emptyset) = 0$, for each $E \in \mathcal{A}$, is defined by*

$$|m|(E) = \sup\{\textstyle\sum_{i=1}^{n} |m(E_i) - m(E_{i-1})| :$$
$$\emptyset = E_0 \subset E_1 \subset \cdots \subset E_n = E,\, E_i \in \mathcal{A}, i = 1, \ldots, n\}.$$

In the previous definition, the supremum is taken over all chain between $\emptyset$ and E.

The chain variation $|m|$ of a set function m is positive, monotone set function, vanishing at the empty set, and the inequality $|m(E)| \leq |m|(E)$ is satisfied for each $E \in \mathcal{A}$. Consequently, if m is a fuzzy measure, then $|m|(E) = m(E)$, for all $E \in \mathcal{A}$.

Definition 3 *A real-valued set function m, $m(\emptyset) = 0$, is of bounded chain variation if $|m|(X) < \infty$.*

The family of all set functions of bounded chain variation, vanishing at the empty set, is denoted by BV. The functional $||m|| = |m|(X)$ is a norm on a Banach space $(BV, ||\ \ ||)$, see [1, 10].

Another important characterization of space BV, given by following theorem, has been proven in [1, 10].

Theorem 1 *A set function m, $m(\emptyset) = 0$, belongs to BV if and only if it can be represented as difference of two monotone set functions m_1 and m_2 vanishing at the empty set.*

Using Theorem 1, another representation of Choquet integral with respect to a signed fuzzy measure can be obtained [10], and this is illustrated in the following example.

Example 1 Let X be a finite set, $X = \{1, 2, 3, 4\}$, $\mathcal{A} = \mathcal{P}(X)$ and let m be a signed fuzzy measure, $m \in BV$, defined by

$m(\{1\})$=0.3	$m(\{2\})$=0.2	$m(\{3\})$=−0.3	$m(\{4\})$=−0.4
$m(\{1,2\})$=1	$m(\{1,3\})$=0.2	$m(\{1,4\})$=−0.2	$m(\{2,3\})$=0.1
$m(\{2,4\})$=−0.4	$m(\{3,4\})$=−1	$m(\{1,2,3\})$=0.5	$m(\{1,2,4\})$=0.2
$m(\{2,3,4\})$=−0.4	$m(\{1,3,4\})$=−0.3	$m(\emptyset)$=0	$m(X)$=0

For $f \in \overline{\mathcal{M}}$ defined by $f(1) = -0.3$, $f(2) = 0.2$, $f(3) = -0.4$ and $f(4) = 0.6$, we compute C_m as

$$C_m(f) = -0.4{\cdot}0+(-0.3+0.4){\cdot}0.2+(0.2+0.3){\cdot}(-0.4)+(0.6-0.2){\cdot}(-0.4) = -0.34,$$

or by the equation (2) we obtain

$$C_m(f) - C_{\bar{m}}(f) = -0.24 - 0.1 = -0.34.$$

However, $m \in BV$, and therefore it can be represented as difference of two fuzzy measures m_1 and m_2, i.e. $m = m_1 - m_2$, (this representation is not unique). m_1 can be defined by

$m_1(\{1\})$=0.3	$m_1(\{2\})$=0.2	$m_1(\{3\})$=0	$m_1(\{4\})$=0
$m_1(\{1,2\})$=1	$m_1(\{1,3\})$=0.5	$m_1(\{1,4\})$=0.3	$m_1(\{2,3\})$=0.4
$m_1(\{2,4\})$=0.2	$m_1(\{3,4\})$=0	$m_1(\{1,2,3\})$=1	$m_1(\{1,2,4\})$=1
$m_1(\{2,3,4\})$=0.6	$m_1(\{1,3,4\})$=0.7	$m_1(\emptyset)$=0	$m_1(X)$=1

and then m_2 is given by

$m_2(\{1\})$=0	$m_2(\{2\})$=0	$m_2(\{3\})$=0.3	$m_2(\{4\})$=0.4
$m_2(\{1,2\})$=0	$m_2(\{1,3\})$=0.3	$m_2(\{1,4\})$=0.5	$m_2(\{2,3\})$=0.3
$m_2(\{2,4\})$=0.6	$m_2(\{3,4\})$=1	$m_2(\{1,2,3\})$=0.5	$m_2(\{1,2,4\})$=0.8
$m_2(\{2,3,4\})$=1	$m_2(\{1,3,4\})$=1	$m_2(\emptyset)$=0	$m_2(X)$=1

Then we have

$$C_{m_1}(f) = -0.4 + (-0.3 + 0.4) \cdot 1 + (0.2 + 0.3) \cdot 0.2 + (0.6 - 0.2) \cdot 0 = -0.2,$$

and

$$C_{m_2}(f) = -0.4 + (-0.3 + 0.4) \cdot 0.8 + (0.2 + 0.3) \cdot 0.6 + (0.6 - 0.2) \cdot 0.4 = 0.14.$$

Therefore we have

$$C_m(f) = C_{m_1}(f) - C_{m_2}(f) = -0.2 - 0.14 = -0.34.$$

Theorem 2 *If m is a signed fuzzy measure such that $m \in BV$, then the asymmetric Choquet integral of $f \in \overline{\mathcal{M}}$ can be represented in the following manner*

$$C_m(f) = C_{m_1}(f) - C_{m_2}(f), \tag{3}$$

where m_1 and m_2 are two fuzzy measures such that $m = m_1 - m_2$ and $C_m(f)$ does not depend of the representation of m by means of Theorem 1.

`Proof.` It follows from Theorem 1 that every signed fuzzy measure $m \in BV$ is a difference of two fuzzy measures m_1 and m_2, i.e., $m = m_1 - m_2$. The conjugate set function $\bar{m}$ is difference of the two corresponding conjugate fuzzy measures $\bar{m}_1$ and $\bar{m}_2$, i.e., $\bar{m} = \overline{m_1 - m_2} = \bar{m}_1 - \bar{m}_2$. By the equation (2) and additivity of integrals we obtain $C_m(f) = C_{m_1}(f) - C_{m_2}(f)$.

We prove now that the integral C_m is independent of the representation of the fuzz measure m. Let $\widetilde{m}_1$ and $\widetilde{m}_2$ be two fuzzy measures such that $m = \widetilde{m}_1 - \widetilde{m}_2$. We have $m = m_1 - m_2 = \widetilde{m}_1 - \widetilde{m}_2$ and $\bar{m} = \bar{m}_1 - \bar{m}_2 = \overline{\widetilde{m}}_1 - \overline{\widetilde{m}}_2$ and it follows from this fact that the equation (3) unambiguously represents $C_m(f)$. □

As we have already seen in Example 1 it is not difficult to construct example to show that the representation of m given by Theorem 1 is not unique but by Theorem 2 $C_m(f)$ does not depend of representation of m.

Example 2 Let m and f be as in Example 1, and $m = \widetilde{m}_1 - \widetilde{m}_2$, where $\widetilde{m}_1$ is defined by

$\widetilde{m}_1(\{1\})=0.3$	$\widetilde{m}_1(\{2\})=0.2$	$\widetilde{m}_1(\{3\})=0$	$\widetilde{m}_1(\{4\})=0$
$\widetilde{m}_1(\{1,2\})=1$	$\widetilde{m}_1(\{1,3\})=0.6$	$\widetilde{m}_1(\{1,4\})=0.5$	$\widetilde{m}_1(\{2,3\})=0.5$
$\widetilde{m}_1(\{2,4\})=0.3$	$\widetilde{m}_1(\{3,4\})=0$	$\widetilde{m}_1(\{1,2,3\})=1$	$\widetilde{m}_1(\{1,2,4\})=1$
$\widetilde{m}_1(\{2,3,4\})=0.6$	$\widetilde{m}_1(\{1,3,4\})=0.7$	$\widetilde{m}_1(\emptyset)=0$	$\widetilde{m}_1(X)=1$

and $\widetilde{m}_2$ is given by

$\widetilde{m}_2(\{1\})=0$	$\widetilde{m}_2(\{2\})=0$	$\widetilde{m}_2(\{3\})=0.3$	$\widetilde{m}_2(\{4\})=0.4$
$\widetilde{m}_2(\{1,2\})=0$	$\widetilde{m}_2(\{1,3\})=0.4$	$\widetilde{m}_2(\{1,4\})=0.7$	$\widetilde{m}_2(\{2,3\})=0.4$
$\widetilde{m}_2(\{2,4\})=0.7$	$\widetilde{m}_2(\{3,4\})=1$	$\widetilde{m}_2(\{1,2,3\})=0.5$	$\widetilde{m}_2(\{1,2,4\})=0.8$
$\widetilde{m}_2(\{2,3,4\})=1$	$\widetilde{m}_2(\{1,3,4\})=1$	$\widetilde{m}_2(\emptyset)=0$	$\widetilde{m}_2(X)=1$

.

Then we have

$$C_{\widetilde{m}_1}(f) = -0.4 + (-0.3 + 0.4) \cdot 1 + (0.2 + 0.3) \cdot 0.3 + (0.6 - 0.2) \cdot 0 = -0.15$$

and

$$C_{\widetilde{m}_2}(f) = -0.4 + (-0.3 + 0.4) \cdot 0.8 + (0.2 + 0.3) \cdot 0.7 + (0.6 - 0.2) \cdot 0.4 = 0.19$$

and finally $C_m(f) = C_{\widetilde{m}_1}(f) - C_{\widetilde{m}_2}(f) = -0.34$, the same value as in Example 1 for $C_m = C_{m_1} - C_{m_2}$.

4 Representation of signed fuzzy measures

In this section we shall consider a representation of a signed fuzzy measure $m : \mathcal{A} \to [-\infty, \infty]$ which belongs to the space BV. We will correspond to it a signed measure μ defined on a σ-algebra $\mathcal{B}$ of subsets of a set Y.

First, we will introduce an interpreter for measurable sets and a frame for representation [5, 8], see [10].

Definition 4 *A mapping $H : \mathcal{A} \to \mathcal{B}$ is called an interpreter if H satisfies*

(i) $H(\emptyset) = \emptyset$ and $H(X) = Y$;

(ii) $H(E) \subset H(F)$, for all $E \subset F$.

A triple $(Y, \mathcal{B}, H)$ is called a frame of $(X, \mathcal{A})$, if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$.

Definition 5 *Let m be a signed fuzzy measure defined on $\mathcal{A}$. A quadruple $(Y, \mathcal{B}, \mu, H)$ is called a representation of m (or $(X, \mathcal{A}, m)$) if H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$, μ is a signed measure on $(Y, \mathcal{B})$, and $m = \mu \circ H$.*

Theorem 3 *Every signed fuzzy measure m, $m \in BV$, has its representation.*

`Proof.` Let m be a signed fuzzy measure, and $m \in BV$. There exist two fuzzy measures, m_1 and m_2, such that $m(E) = m_1(E) - m_2(E)$ for all $E \in \mathcal{A}$. We take for Y the open interval $(-m_2(X), m_1(X))$, and for $\mathcal{B}$ the class of all Borel subsets of Y. We define the mapping $H : \mathcal{A} \to \mathcal{B}$ by $H(E) = (-m_2(E), m_1(E))$, for all $E \in \mathcal{A}$. Then we have $H(\emptyset) = \emptyset$ and $H(X) = (-m_2(X), m_1(X)) = Y$. For $E \subset F$ we have $m_1(E) \leq m_1(F)$ and $m_2(E) \leq m_2(F)$, and therefore $H(E) = (-m_2(E), m_1(E)) \subset (-m_2(F), m_1(F)) = H(F)$. Therefore H is an interpreter from $\mathcal{A}$ to $\mathcal{B}$.

Let $\mu : \mathcal{B} \to \mathbb{R}$ be a signed measure defined by

$$\mu((a, b)) = \lambda((a, b) \cap Y^+) - \lambda((a, b) \cap Y^-), \quad \text{for} \quad (a, b) \in \mathcal{B},$$

where λ is the Lebesgue measure and $Y^+ = (0, m_1(X))$, $Y^- = Y \setminus Y^+$. Hence for every $E \in \mathcal{A}$

$$\begin{aligned} m(E) &= m_1(E) - m_2(E) \\ &= \lambda((0, m_1(E))) - \lambda((-m_2(E), 0]) \\ &= \mu(H(E)) \\ &= \mu \circ H(E). \end{aligned}$$

Therefore, $(Y, \mathcal{B}, \mu, H)$ is a representation of m. □

Remark 1 (i) As it is mentioned before, two fuzzy measures m_1 and m_2 are not unique, hence the representation of m given in Theorem 3 is *not unique*, too.

(ii) If m is a signed fuzzy measure such that $m \in BV$, and $\bar{m}$ is its conjugate set function, then a quadruple $(Y, \mathcal{B}, \mu, \bar{H})$ is a representation of $\bar{m}$, where the interpreter $\bar{H}$ is defined by $\bar{H}(E) = (-\bar{m}_2(E), \bar{m}_1(E))$ for all $E \in \mathcal{A}$, and $(Y, \mathcal{B}, \mu)$ is the same as in the proof of Theorem 3.

Now, we can apply Theorem 3 to obtain a representation of the asymmetric Choquet integral of a measurable function f with respect to a signed fuzzy measure m.

Theorem 4 *If m is a signed fuzzy measure, $m \in BV$ and $f \in \overline{\mathcal{M}}$, then there exist two functions $I_f^1 : Y \to [0, \infty]$ and $I_f^2 : Y \to [0, \infty]$ such that the asymmetric Choquet integral has the following difference representation*

$$C_m(f) = \int I^1_{f^+}\, d\mu - \int I^2_{f^-}\, d\mu, \tag{4}$$

where, $f^+ = f \vee 0$, $f^- = (-f) \vee 0$ and the integrals on the right-hand side are the Lebesgue integrals. $C_m(f)$ does not depend of the representation of m by means of Theorem 3.

`Proof.` Let m be a signed fuzzy measure and let $\bar{m}$ be its conjugate set function, with representations $(Y, \mathcal{B}, \mu, H)$ and $(Y, \mathcal{B}, \mu, \bar{H})$, respectively. If we define for a non-negative measurable function $f \in \mathcal{M}$, two functions I_f^1 and I_f^2 on Y in the following way

$$I_f^1(y) = \sup\{\, t \,|\, y \in H(\{x | f(x) \geq t\})\},$$

$$I_f^2(y) = \sup\{\, t \,|\, y \in \bar{H}(\{x | f(x) \geq t\})\, \}$$

for all $y \in Y$, then $C_m(f^+) = \int I^1_{f^+}\, d\mu$ and $C_{\bar{m}}(f^-) = \int I^2_{f^-}\, d\mu$. Therefore the equation (4) immediately follows from the equation (2).

Let $\widetilde{m}_1$ and $\widetilde{m}_2$ be two fuzzy measures such that $m = \widetilde{m}_1 - \widetilde{m}_2$, and $(\widetilde{Y}, \mathcal{B}, \mu, \widetilde{H})$ and $(\widetilde{Y}, \mathcal{B}, \mu, \overline{\widetilde{H}})$ are the representations of m and $\bar{m}$, where $\widetilde{H}$ and $\overline{\widetilde{H}}$ are the interpreters defined by: $\widetilde{H}(E) = (-\widetilde{m}_2(E), \widetilde{m}_1(E))$ and $\overline{\widetilde{H}}(E) = (-\overline{\widetilde{m}}_2(E), \overline{\widetilde{m}}_1(E))$, for all $E \in \mathcal{A}$. We have $m = \mu \circ H = \mu \circ \widetilde{H}$ and $\bar{m} = \mu \circ \overline{H} = \mu \circ \overline{\widetilde{H}}$. $\widetilde{I}_1(f)$ and $\widetilde{I}_2(f)$ are defined on $\widetilde{Y}$ by

$$\widetilde{I}_1(f)(y) = \sup\{\, t \,|\, y \in \widetilde{H}(\{x | f(x) \geq t\})\} \qquad \text{and}$$

$$\widetilde{I}_2(f)(y) = \sup\{\, t \,|\, y \in \overline{\widetilde{H}}(\{x | f(x) \geq t\})\, \} \text{ for all } y \in \widetilde{Y}.$$

Therefore for every $f \in \overline{\mathcal{M}}$

$$C_m(f^+) = \int I_1(f^+)\, d\mu = \int \widetilde{I}_1(f^+)\, d\mu \text{ and}$$

$$C_{\bar{m}}(f^-) = \int I_2(f^-)\, d\mu = \int \widetilde{I}_2(f^-)\, d\mu.$$

Hence the equation (4) unambiguously represents $C_m(f)$. □

Example 3 Let m and f be defined same as in the Example 1.

Let $(Y, \mathcal{B}, \mu, H)$ and $(Y, \mathcal{B}, \mu, \bar{H})$ be the representations of m and $\bar{m}$ related to m_1 and m_2 given in Example 1. Therefore $Y = (-1, 1)$ and we have

$$I_1(f^+)(y) = \begin{cases} 0, & y \in (-1, -0.6] \cup [0.2, 1) \cup \{0\} \\ 0.2, & y \in (0, 0.2) \cup (-0.6, -0.4) \\ 0.6, & y \in (-0.4, 0), \end{cases}$$

and

$$I_2(f^-)(y) = \begin{cases} 0, & y \in (-1, -0.4] \cup [0.8, 1) \cup \{0\} \\ 0.3, & y \in (0, 0.8) \cup (-0.4, -0.2) \\ 0.4, & y \in (-0.2, 0). \end{cases}$$

We have $\int I_1(f^+)\, d\mu - \int I_2(f^-)\, d\mu = -0.24 - 0.1 = -0.34 = C_m(f)$.

Now we consider the representation of m, $(\widetilde{Y}, \mathcal{B}, \mu, \widetilde{H})$, related to $\widetilde{m}_1$ and $\widetilde{m}_2$ given in Example 2 and the appropriate representation of $\bar{m}$. Then $\widetilde{Y} = (-1, 1)$ and we have

$$\widetilde{I}_1(f^+)(y) = \begin{cases} 0, & y \in (-1, -0.7] \cup [0.3, 1) \cup \{0\} \\ 0.2, & y \in (0, 0.3) \cup (-0.7, -0.4) \\ 0.6, & y \in (-0.4, 0), \end{cases}$$

and

$$\widetilde{I}_2(f^-)(y) = \begin{cases} 0, & y \in (-1, -0.3] \cup [0.7, 1) \cup \{0\} \\ 0.3, & y \in (0, 0.7) \cup (-0.3, -0.2) \\ 0.4, & y \in (-0.2, 0). \end{cases}$$

We compute $\int \widetilde{I}_1(f^+)\, d\mu = -0.24$ and $\int \widetilde{I}_2(f^-)\, d\mu = 0.1$ and applying (4), finally we obtain $C_m(f) = -0.34$.

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