Weak Convergence of Random Sets

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Abstract: In this paper the classical Portmanteau theorem which provides equivalent conditions of weak convergence of sequence of probability measures is extended on the space of the sequence of probability measures induced by random sets.

Key words and phrases: Weak convergence, Random Set, Probability measure, Probability space, Distribution function, Capacity functional, Inclusion functional.

1 Introduction

Random sets have been used as a generalization of random variables, [5, 7]. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space. While random variables associate to elements of Ω single elements of $\mathbb{R}$, random sets associate nonempty subsets of $\mathbb{R}$ to elements of Ω.

Research of weak convergence of sequence random sets follows from weak convergence of random variables. Weak convergence is very important in probability theory. It has been used in central limit theorems, [1, 2], specially by large deviation principle. The theory of large deviations concerned with the asymptotic estimation of probabilities of rare events, and typically provide exponential bound on probability of such events and characterize them. This theory has found many applications in information theory, coding theory, image processing, statistical mechanics, various kind of random processes (certain types of finite state Markov chains, Brownian motion, Wiener process), stochastic differential equations, etc., see [3]. The Theorem of Portmanteau establishes some equivalent statements for large deviation convergence. In this paper we prove a Portmanteau type theorem for random sets.

The paper is organized as follows. Section 2 contains some basic definitions and theorems from probability theory. Section 3 gives some equivalent conditions of weak convergence (connections between convergence of sequence of random variables, convergence of sequence of probability measures and convergence of sequence of distribution functions) and Portmanteau theorem. In Section 4 we present some basic notions and definitions of random sets. The main result of this paper is Portmanteau theorem for random sets and it is proved in Section 5.

2 Preliminaries

Let Ω be an arbitrary space or set of points ω. Set Ω contains all possible results of an experiment. A class $\mathcal{A}$ of subsets of Ω is called *σ-algebra* if it contains Ω and is closed under the formation of complements and countable unions. A *set function* is a real valued function defined on some class subsets of Ω.

Definition 1 *A set function* P *on a σ-algebra $\mathcal{A}$ is a* probability measure *if it satisfies these conditions:*

(i) $0 \leq \mathsf{P}(A) \leq 1$ *for every* $A \in \mathcal{A}$*;*

(ii) $\mathsf{P}(\emptyset) = 0, \ \ \mathsf{P}(\Omega) = 1$*;*

(iii) *if* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of pairwise disjoint sets from* $\mathcal{A}$*, then*

$$\mathsf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathsf{P}(A_n).$$

The next theorem gives some useful well-known properties of probability measures (see [1, 2]).

Theorem 1 *Let* P *be a probability measure on a σ-algebra* $\mathcal{A}$*.*

(i) Continuity from below*: If* $(A_n)_{n\in\mathbb{N}}$ *and A lie in $\mathcal{A}$ and* $A_n \uparrow A$*, then* $\mathsf{P}(A_n) \uparrow \mathsf{P}(A)$.

(ii) Continuity from above*: If* $(A_n)_{n\in\mathbb{N}}$ *and A lie in $\mathcal{A}$ and* $A_n \downarrow A$*, then* $\mathsf{P}(A_n) \downarrow \mathsf{P}(A)$.

(iii) Monotonicity*: If* $A \subset B$*, then* $\mathsf{P}(A) \leq \mathsf{P}(B)$*, for* $A, B \in \mathcal{A}$*.*

Definition 2 *A measure space* $(\Omega, \mathcal{A}, \mathsf{P})$ *is called* probability space*.*

Definition 3 *Let* $(\Omega, \mathcal{A}, \mathsf{P})$ *be an arbitrary probability space, and let* X *be a real-valued function on* Ω *such that for all* $x \in \mathbb{R}$

$$\{\omega \mid \omega \in \Omega, \mathsf{X}(\omega) < x\} \in \mathcal{A}.$$

Then X *is a* random variable *on probability space* $(\Omega, \mathcal{A}, \mathsf{P})$*.*

It is clear that a random variable $\mathsf{X} : \Omega \to \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ generates a *probability distribution* P_X, for every $B \in \mathcal{B}$ ($\mathcal{B}$ is Borel σ-algebra subsets of $\mathbb{R}$), defined with

$$\mathsf{P}_\mathsf{X}(B) = \mathsf{P}(\{\omega \mid \mathsf{X}(\omega) \in B\}) = \mathsf{P}(\mathsf{X}^{-1}(B)).$$

It is easy to check that P_X satisfies conditions of Definition 1, i.e., P_X is a probability measure, i.e., *probability measure induced by random variable.*

Definition 4 *The* distribution function *of the random variable X is the function* $\mathsf{F}_\mathsf{X}(x) = \mathsf{P}(\mathsf{X} \leq x), \quad x \in \mathbb{R}$.

The next theorem gives relations between probability measure P_X and distribution function F_X (see [1, 2]).

Theorem 2 (i) *Let* P_X *be a probability measure defined on measurable space* $(\mathbb{R}, \mathcal{B})$ *and let* $\mathsf{F}_\mathsf{X} : \mathbb{R} \to \mathbb{R}$ *be a function defined by*

$$\mathsf{F}_\mathsf{X}(x) = \mathsf{P}_\mathsf{X}((-\infty, x]). \tag{1}$$

Then F_X *is a distribution function.*

(ii) *If* F_X *is a distribution function, then exists a unique probability measure* P_X *defined on a measurable space* $(\mathbb{R}, \mathcal{B})$*, such that for every real* x (1) *holds.*

In probability theory four different types of convergence are considered : convergence almost everywhere, convergence in the mean, convergence in probability and convergence in distribution (weak convergence). We are interested in weak convergence of random variables and we need the following definitions: convergence of sequence of distribution functions and convergence of sequence of probability measures (see [1, 2]).

Definition 5 *Sequence of distribution functions* $(\mathsf{F}_n)_{n\in\mathbb{N}}$ converges weakly *to the distribution function* F*, (write* $\mathsf{F}_n \Rightarrow \mathsf{F}$*), if*

$$\lim_{n\to\infty} \mathsf{F}_n(x) = \mathsf{F}(x) \tag{2}$$

for every continuity point x *of* F.

Definition 6 *Let* $\mathsf{P}, \mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3, \ldots$ *be probability measures defined on measurable space* $(\mathbb{R}, \mathcal{B})$*. Sequence of probability measures* $(\mathsf{P}_n)_{n\in\mathbb{N}}$ converges weakly *to the probability measure* P*,(write* $\mathsf{P}_n \Rightarrow \mathsf{P}$*), if for every bounded and continuous function* $f : \mathbb{R} \to \mathbb{R}$ *holds*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f \;\; d\mathsf{P}_n = \int_{\mathbb{R}} f \;\; d\mathsf{P}. \tag{3}$$

Definition 7 *The* expected value *of a real variable* X *is the Lebesgue integral function* X *with respect to measure* P

$$\mathsf{E}(\mathsf{X}) = \int_{\Omega} \mathsf{X} \ \ d\mathsf{P}. \tag{4}$$

If $h : \mathbb{R} \to \mathbb{R}$ is a Borel's function, then $h(\mathsf{X})$ is a random variable too, and

$$\mathsf{E}(h(\mathsf{X})) = \int_{\Omega} h(\mathsf{X}) \ \ d\mathsf{P}.$$

3 Portmanteau Theorem

In this section we observe random variables $\mathsf{X}, \mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3, \ldots$ defined on the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

Definition 8 *A sequence of random variables* $(\mathsf{X}_n)_{n\in\mathbb{N}}$ converges in distribution *(or* converges weakly*) to the random variable* X, *(write* $\mathsf{X}_n \xrightarrow{\mathcal{D}} \mathsf{X}$*) if for every bounded and continuous function* $f : \mathbb{R} \to \mathbb{R}$ *holds*

$$\lim_{n\to\infty} \mathsf{E}(f(\mathsf{X}_n)) = \mathsf{E}(f(\mathsf{X})). \tag{5}$$

The following theorems provide useful conditions equivalent to weak convergence (see [1]). A set A in $\mathcal{B}$ is called a P-*continuity set* if boundary ∂A satisfies $\mathsf{P}(\partial A) = 0$.

Theorem 3 *Let* $\mathsf{X}, \mathsf{X}_1, \mathsf{X}_2, \ldots$ *be random variables, all defined on the same probability space* $(\Omega, \mathcal{A}, \mathsf{P})$. *Let* $\mathsf{F}, \mathsf{F}_1, \mathsf{F}_2, \ldots$ *be their distribution function and* $\mathsf{P}, \mathsf{P}_1, \mathsf{P}_2, \ldots$ *are probability measures on* $(\mathbb{R}, \mathcal{B})$ *corresponding to* $\mathsf{F}, \mathsf{F}_1, \mathsf{F}_2, \ldots$. *The following conditions are equivalent:*

(i) *Sequence of random variables* $(\mathsf{X}_n)_{n\in\mathbb{N}}$ *converges in distribution to the* X.

(ii) *Sequence of distribution function* $(\mathsf{F}_n)_{n\in\mathbb{N}}$ *converges weakly to the* F.

(iii) *Sequence of probability measures* $(\mathsf{P}_n)_{n\in\mathbb{N}}$ *converges weakly to the* P.

Theorem 4 (Portmanteau) *Let* P, P_n *(*$n \in \mathbb{N}$*) be probability measures on* $(\mathbb{R}, \mathcal{B})$. *These five conditions are equivalent:*

(i) $\mathsf{P}_n \Rightarrow \mathsf{P}$.

(ii) $\lim_{n\to\infty} \int_{\mathbb{R}} f \ \ d\mathsf{P}_n = \int_{\mathbb{R}} f \ \ d\mathsf{P}$ *for every bounded, uniformly continuous real functions* f.

(iii) $\limsup_{n} \mathsf{P}_n(F) \leq \mathsf{P}(F)$ *for every closed set* F.

(iv) $\liminf_n \mathsf{P}_n(G) \geq \mathsf{P}(G)$ *for every open set* G.

(v) $\lim_{n\to\infty} \mathsf{P}_n(A) = \mathsf{P}(A)$ *for every* P*-continuity set* A.

4 Random Sets

A random closed set is a random element in the space of all closed subsets of the real line. Let $\mathcal{O}$, $\mathcal{F}$, $\mathcal{K}$ and $\mathcal{C}$ denote the collection of open subsets, closed subsets, compact subsets and convex compact subsets of $\mathbb{R}$, respectively. We introduce sub-classes $\mathcal{F}^K$ and $\mathcal{F}_G$ of $\mathcal{F}$ by

$$\mathcal{F}^K = \{F \in \mathcal{F} \mid F \cap K = \emptyset\}, \quad \text{for} \quad K \in \mathcal{K},$$
$$\mathcal{F}_G = \{F \in \mathcal{F} \mid F \cap G \neq \emptyset\}, \quad \text{for} \quad G \in \mathcal{O}.$$

The collections $\{\mathcal{F}^K \mid K \in \mathcal{K}\}$ and $\{\mathcal{F}_G \mid G \in \mathcal{O}\}$ generate a topology on $\mathcal{F}$. This topology is known as the *hit-or-miss-topology.* Class $\mathcal{F}$ is endowed with Borel σ-algebra $\Sigma(\mathcal{F})$ generated by hit-or-miss-topology (see [7]).

Definition 9 *A* random closed set S *is a measurable transformation from a probability space* $(\Omega, \mathcal{A}, \mathsf{P})$ *into the measurable space* $(\mathcal{F}, \Sigma(\mathcal{F}))$.

Random closed set S generates a *probability distribution* P_S, for every $A \in \Sigma(\mathcal{F})$ in the following way

$$\mathsf{P}_\mathrm{S}(A) = \mathsf{P}(\{\omega \in \Omega \mid \mathrm{S}(\omega) \in A\}) = \mathsf{P}_\mathrm{S}(\mathrm{S} \in A). \tag{6}$$

In the theory of random sets all information about a random set is contained in the capacity functional of S.

Definition 10 *The* capacity functional $\mathrm{T}_\mathrm{S}(K)$ *(*$K \in \mathcal{K}$*) of random closed set* S *is defined by*

$$\mathrm{T}_\mathrm{S}(K) = \mathsf{P}_\mathrm{S}(\mathrm{S} \in \mathcal{F}_K) = \mathsf{P}_\mathrm{S}(\mathrm{S} \cap K \neq \emptyset). \tag{7}$$

In the theory of random sets, capacity functional plays the same role as distribution function in probability theory. More about capacity functional and properties can be found in [7] and [5]. Convex random closed sets and convex compact random sets have a very important role in the theory of random sets. Let $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{K}$ be the class of all convex compact subsets of $\mathbb{R}$.

Definition 11 *A random closed set* S *is said to be* convex *if its realizations are almost surely convex, i.e.* $\mathrm{S}(\omega)$ *belongs to* $\mathcal{C}$ *almost surely.*

Definition 12 *A random closed set* S *is said to be* convex compact *if it* $\mathrm{S}(\omega)$ *belongs to* $\mathcal{C}_0$ *almost surely.*

The distribution of any convex compact random set is determined by the inclusion functional of S, (see [8]).

Definition 13 *The* inclusion functional $\mathrm{t_S}(K)$ *$(K \in \mathcal{C}_0)$ of convex compact random set* S *is defined by*

$$\mathrm{t_S}(K) = \mathsf{P}_\mathrm{S}(\mathrm{S} \subset K). \tag{8}$$

The functional $\mathrm{T_S}(K), \quad K \in \mathcal{C}_0$ is naturally extended onto the class $\mathcal{C}$ by $\mathrm{t_S}(F) = \mathsf{P}_\mathrm{S}(\mathrm{S} \subset F), \;\; F \in \mathcal{C}$.

5 Portmanteau Theorem for Random Sets

Weak convergence of random sets is actually weak convergence of sequence of probability distributions induced by sequence of random sets (see [8]).

Definition 14 *A sequence of random closed sets* $(\mathrm{S}_n)_{n\in\mathbb{N}}$ converges weakly *if the corresponding sequence of probability measures* $(\mathsf{P}_n)_{n\in\mathbb{N}}$ *(generated by random closed sets) converges weakly in the usual sense, i.e.,*

$$\mathsf{P}_n(A) \to \mathsf{P}(A) \tag{9}$$

for each $A \in \Sigma(\mathcal{F})$ *such that* $\mathsf{P}(\partial A) = 0$.

Weak convergence of sequence of probability measures will be denoted in the usual way, $\mathsf{P}_n \Rightarrow \mathsf{P}$. We observe that

$$\mathcal{S}_\mathrm{T} = \{K \in \mathcal{K} \mid \mathrm{T_S}(K) = \mathrm{T_S}(\mathrm{int}K)\},$$

where $\mathrm{int}K$ is the interior of set K. Pointwise convergence of capacity functionals on $\mathcal{S}_\mathrm{T}$ implies the weak convergence of the corresponding probability measures on $\Sigma(\mathcal{F})$ ([8]). The following results are due to Molchanov, see ([8]).

Theorem 5 *A sequence of convex compact random sets* $(\mathrm{S}_n)_{n\in\mathbb{N}}$ *converges weakly to a random closed set* $\tilde{\mathrm{S}}$ *if for every* $K \in \mathcal{C}_0$ *holds*

$$\mathrm{t}_n(K) \to \tilde{\mathrm{t}}(K), \quad as \;\; n \to \infty,$$

where $\mathrm{t}_n, \tilde{\mathrm{t}}$ *are the inclusion functionals of random sets* S_n *and* $\tilde{\mathrm{S}}$ *respectively.*

Let $\mathcal{C}_t = \{F \in \mathcal{C}_0 \mid \mathrm{t}(F) = \mathrm{t}(\mathrm{int}(F))\}$.

Theorem 6 *A sequence of convex compact random sets* $(\mathrm{S}_n)_{n\in\mathbb{N}}$ *converges weakly to a random closed set* $\tilde{\mathrm{S}}$ *if for every* $K \in \mathcal{C}_t$ *holds* $\mathrm{t}_n(K) \to \tilde{\mathrm{t}}(K), \quad as \;\; n \to \infty$, *where* $\mathrm{t}_n, \tilde{\mathrm{t}}$ *are the inclusion functionals of random sets* S_n *and* $\tilde{\mathrm{S}}$ *respectively.*

The next result shows equivalent conditions for weak convergence of sequence probability measures induced by random closed sets.

Theorem 7 (Portmanteau) *Let* P, P_n *$(n \in \mathbb{N})$ be probability measures on* $(\mathcal{F}, \Sigma(\mathcal{F}))$. *The following three conditions are equivalent:*

(i) $\mathsf{P}_n \Rightarrow \mathsf{P}$.

(ii) $\limsup_n \mathsf{P}_n(F) \leq \mathsf{P}(F)$ *for every closed set* F.

(iii) $\liminf_n \mathsf{P}_n(G) \geq \mathsf{P}(G)$ *for every open set* G.

Proof

(i) $\Rightarrow$ (ii) For a given sequence $\delta_k > 0$, $\lim_{k\to\infty} \delta_k = 0$, we construct a sequence of closed sets F_k such that $F \subset F_k$, $F_k \downarrow F$ and $\mathsf{P}(\partial F_k) = 0$. Since (i) holds, $F \subset F_k$ and $\mathsf{P}(\partial F_k) = 0$, then

$$\limsup_n \mathsf{P}_n(F) \leq \limsup_n \mathsf{P}_n(F_k) = \mathsf{P}(F_k)$$

for each k.

F is closed and $F_k \downarrow F$ so (ii) follows.

(ii) $\Rightarrow$ (iii) Let G be an arbitrary open set. Then the complement of G, denoted by G^c, is a closed set and using (ii) and $\mathsf{P}_n(G) + \mathsf{P}_n(G^c) = 1$ we have

$$\begin{aligned} 1 - \mathsf{P}(G) &= \mathsf{P}(G^c) \geq \limsup_n \mathsf{P}_n(G^c) \\ &= \limsup_n (1 - \mathsf{P}_n((G^c)^c)) \\ &= \limsup_n (1 - \mathsf{P}_n(G)) \\ &= 1 - \liminf_n \mathsf{P}_n(G). \end{aligned}$$

We have $1 - \mathsf{P}(G) \geq 1 - \liminf_n \mathsf{P}_n(G)$ and (iii) follows.

(iii) $\Rightarrow$ (ii) The proof is analogous to the proof of (ii) $\Rightarrow$ (iii).

(ii) $\wedge$ (iii) $\Rightarrow$ (i) Let $\mathrm{int}A$ denote the interior of set A, and let $\mathrm{cl}A$ denote its closure. If A is a set such that $\mathsf{P}(\partial A) = 0$, than $\mathsf{P}(\mathrm{cl}A) = \mathsf{P}(\mathrm{int}A)$.

From (i) follows $\limsup_n \mathsf{P}_n(\mathrm{cl}A) \leq \mathsf{P}(\mathrm{cl}A) = \mathsf{P}(\mathrm{int}A)$.

From (iii) follows $\liminf_n \mathsf{P}_n(\mathrm{int}A) \geq \mathsf{P}(\mathrm{int}A)$.

And we have

$$\limsup_n \mathsf{P}_n(\mathrm{cl}A) \leq \mathsf{P}(\mathrm{int}A) \leq \liminf_n \mathsf{P}_n(\mathrm{int}A). \tag{10}$$

From $\mathrm{int}B \subset \mathrm{cl}B$ and the monotonicity of P_n follows

$$\limsup_n \mathsf{P}_n(\mathrm{int}A) \leq \limsup_n \mathsf{P}_n(\mathrm{cl}A). \tag{11}$$

From (10) and (11) follows

$$\limsup_n \mathsf{P}_n(\mathrm{int}A) \leq \mathsf{P}(\mathrm{int}A) \leq \liminf_n \mathsf{P}_n(\mathrm{int}A). \tag{12}$$

From the definition of lim sup and lim inf follows

$$\liminf_n \mathsf{P}_n(\mathrm{int}A) \leq \limsup_n \mathsf{P}_n(\mathrm{int}A). \tag{13}$$

And finally, from (12) and (13) follows (iii).

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