Copulas as aggregation operators

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Abstract: This paper give some motivations for introducing copulas and presents some recent results on them. There is presented an application of copulas in the theory of aggregation operators. Transformations of copulas by means of increasing bijections on the unit interval and attractors of copulas are discussed. There is presented a result on an approximation of associative copulas by strict and nilpotent triangular norms.

Key words and phrases: Copula, aggregation operator, fuzzy measure, transformation of copulas, maximum attractor, Archimax copula .

1 Introduction

In this paper we give the motivation for introducing a special class of real monotone operations-copulas and we stress the important role of them in many fields. We will present some recent results on copulas, [13, 14, 15, 16, 17]. In Section 2 we present some basic facts about copulas and some their applications. In Section 3 we present an application of copulas for the construction of aggregation operators. In Section 4 transformations of copulas by means of increasing bijections on the unit interval and attractors of copulas are discussed. The invariance of copulas under such transformations as well as the relationship to maximum attractors and Archimax copulas is investigated. In Section 5 we show that the both the strict and the non-strict Archimedean copulas form dense subclasses of the class of associative copulas.

2 Copulas

The well-known Sklar's Theorem [25] states that each random vector (X, Y) is characterized by some copula C in a way that for its joint distribution H_{XY} and for the corresponding marginal distributions F_X and F_Y we have $H_{XY}(x, y) = C(F_X(x), F_Y(y))$.

2.1 Definition
A *(two-dimensional) copula* is a function $C\colon [0,1]^2 \to [0,1]$ such that $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in [0,1]$, and C is 2-increasing, i.e., for all $x, x^*, y, y^* \in [0,1]$ with $x \leq x^*$ and $y \leq y^*$ for the volume Vol_C of the rectangle $[x,x^*] \times [y,y^*]$ we have

$$\mathrm{Vol}_C([x,x^*] \times [y,y^*]) = C(x,y) - C(x,y^*) + C(x^*,y^*) - C(x^*,y) \geq 0.$$

Important examples of copulas are (we shall use the notations from [12]) *the upper bound* triangular norm $T_{\mathbf{M}}$ given by $T_{\mathbf{M}}(x,y) = \min(x,y)$ and *the lower bound* triangular norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, and the *product* $T_{\mathbf{P}}$ given by $T_{\mathbf{P}}(x,y) = x \cdot y$. So we have for every copula C that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$. Copulas form a subclass of the class $\mathcal{V}$ of functions $V\colon [0,1]^2 \to [0,1]$ which are continuous, non-decreasing in each component and satisfy $\mathrm{Ran}\, V = [0,1]$ (the elements of $\mathcal{V}$ are also called *binary aggregation operators* [1, 18]).

We mention here some applications of copulas. Under a.s.strictly increasing transformations of X and Y the copula C_{XY} is invariant, although we can change the margins. Thus (for random variables with continuous distribution functions) the study of rank statistics may be characterized as the study of copulas and copula-invariant properties. For random variables with continuous distribution functions, the extreme copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are attained precisely when X is a.s. an increasing (respectively, decreasing) function of Y. Therefore copulas can be used to construct non-parametric measures of dependence.

Let $\star$ be the binary operation defined on the set of two-dimensional copulas for copulas C_1 and C_2 by

$$C_1 \star C_2(x,y) = \int_0^1 \frac{\partial C_1(x,t)}{\partial t} \cdot \frac{\partial C_2(t,y)}{\partial t}\, dt, \qquad (1)$$

(these partial derivatives exist almost everywhere). Then $C_1 \star C_2$ is a copula, and the set of copulas is a non-commutative semigroup under the operation $\star$. The strong interpretation in the context of Markov processes is the following: If $(X_t)_{t \in I}$ is a real stochastic process with parameter set I and if C_{st} is the copula of X_s and X_t, then the transition probabilities of the process satisfy the Kolmogorov–Chapman equation if and only if $C_{st} = C_{su} \star C_{ut}$ for all $s,t,u \in I$ with $s < u < t$, see [20].

In the theory of probabilistic metric spaces it is important to obtain a rich source of different triangle functions which would enable the construction of new probabilistic metric spaces, see [10, 24]. A *triangle function* τ is a binary operation on the family of all distance distribution functions Δ^+ that is commutative, associative, and non-decreasing in each place, and has H_0 as identity. One of the useful constructions goes in the following way. Let C be a copula and let $L : [0,\infty]^2 \to [0,\infty]$ be a surjective and continuous function on $[0,\infty]^2 \setminus \{(0,\infty),(\infty,0)\}$ and for each $x \in [0,\infty[$ the set $L_x = \{(u,v) \in [0,\infty]^2 \mid L(u,v) < x\}$ is bounded and $([0,\infty], L, \leq)$ is a partially ordered semigroup. The function $\sigma_{C,L} : \Delta^+ \times \Delta^+ \to \Delta^+$ is defined by

$$\sigma_{C,L}(F,G)(x) = \begin{cases} 0 & \text{if } x \in [-\infty, 0], \\ \int_{L_x} dC(F(u), G(v)) & \text{if } x \in (0, \infty), \\ 1 & \text{if } x = \infty, \end{cases}$$

where the integral is of Lebesgue–Stieltjes type. $\sigma_{C,L}$ is a triangle function if and only if $C = (< (a_\alpha, e_\alpha), T_{\mathbf{P}} >)_{\alpha \in A}$, see [24], Corollary 7.4.4.

2.2 Remark

The concept of a copula can be extended to n dimensions. An n-copula is an n-dimensional distribution function whose support is in the unit n-cube and whose one-dimensional margins are uniform, see [20, 24]. If J is an n-dimensional distribution function with one-dimensional margins $F_1, \ldots, F_n$, then there is an n-copula C such that

$$J(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Moreover, for any n-copula:

$$T_{\mathbf{L}}(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq T_{\mathbf{M}}(x_1, \ldots, x_n).$$

The upper function $T_{\mathbf{M}}$ is an n-copula for any $n \in \mathbb{N}$, the lower function $T_{\mathbf{L}}$ is not an n-copula for any $n > 2$. The main problem in the theory of copulas is to determine which sets of copulas (of possible different dimensions) can appear as margins of a single higher-dimensional copula.

3 Aggregation operator construction based on copulas

We present in this section some results from [15]. we remark that such a copula-based approach to aggregation was originally proposed in [11] for the Frank family of t-norms (see, e.g., [7, 12]). Let X be a non-empty index set and $f : X \longrightarrow [0,1]$ the input system to be aggregated. Let $(X, \mathcal{A}, m)$ be a fuzzy measure space, i.e., $\mathcal{A}$ is a σ-algebra of subsets of X (in the case of a finite set X we usually take $\mathcal{A} = 2^X$), and $m : \mathcal{A} \longrightarrow [0,1]$ a fuzzy measure, thus satisfying $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subseteq B$. Denote by $\mathcal{L}(\mathcal{A})$ the set of all $\mathcal{A}$-measurable functions from X to $[0,1]$.

3.1 Definition

Consider two fuzzy measure spaces $(X, \mathcal{A}, m)$ and $(]0,1[^2, \mathcal{B}(]0,1[^2), \mu)$. The functional $M_{m,\mu} : \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$ given by

$$M_{m,\mu}(f) = \mu(D_{m,f}),$$

will be called (m,μ)-*aggregation operator*, where

$$D_{m,f} = \{(x,y) \in]0,1[^2 \mid y < m(\{f \geq x\})\}.$$

Special fuzzy measures μ imply reasonable properties of the (m, μ)-aggregation operator $M_{m,\mu}$:

3.2 Proposition
Let $C\colon [0,1]^2 \longrightarrow [0,1]$ be a copula and denote by μ_C the unique probability measure on $(]0,1[^2, \mathcal{B}(]0,1[^2))$ with $\mu_C(]0,x[\times]0,y[) = C(x,y)$ for all $(x,y) \in]0,1[^2$. Then, for each fuzzy measure space $(X, \mathcal{A}, m)$, the (m, μ_C)-aggregation operator M_{m,μ_C} is an idempotent aggregation operator and we have $M_{m,\mu_C}(\mathbf{1}_A) = m(A)$ for all $A \in \mathcal{A}$.

Choosing adequate copulas C, we obtain some well-known types of integrals.

3.3 Example

(i) If C equals the standard product $T_{\mathbf{P}}$, i.e., $\mu_{T_{\mathbf{P}}}$ is the Lebesgue measure on $\mathcal{B}(]0,1[^2)$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ is just the Choquet integral with respect to m (see [3, 21]). If, in addition, m is a σ-additive measure on $(X, \mathcal{A})$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ coincides with the classical Lebesgue integral with respect to m, and for $X = \{1, 2, \ldots, n\}$ we obtain a weighted mean. If $X = \{1, 2, \ldots, n\}$ and if m is a symmetric fuzzy measure on $(X, 2^X)$ then $M_{m,\mu_{T_{\mathbf{P}}}}$ is an OWA operator.

(ii) If C equals the minimum $T_{\mathbf{M}}$ then

$$\mu_{T_{\mathbf{M}}}(A) = \lambda(\{x \in]0,1[\mid (x,x) \in A\}),$$

and $M_{m,\mu_{T_{\mathbf{M}}}}$ equals the Sugeno integral, see [21]. If $X = \{1, 2, \ldots, n\}$ and if m is a symmetric fuzzy measure on $(X, 2^X)$ then $M_{m,\mu_{T_{\mathbf{M}}}}$ is an WOWM (weighted ordered weighted maximum) operator.

(iii) If C equals the Łukasiewicz t-norm $T_{\mathbf{L}}$ then

$$\mu_{T_{\mathbf{L}}}(A) = \lambda(\{x \in]0,1[\mid (x, 1-x) \in A\}),$$

and if the index set X is finite, then $M_{m,\mu_{T_{\mathbf{L}}}}$ is the so-called opposite Sugeno integral [11].

Related dual aggregation operators we have the following result.

3.4 Proposition
Let X be a finite set. Keeping the notations and hypotheses of Proposition 3.2, we have

$$M^d_{m,\mu_C} = M_{m^d,\mu_{\hat{C}}}. \tag{2}$$

If a copula C coincides with its survival copula $\hat{C}$, then a special form of (2) holds, namely, $M^d_{m,\mu_C} = M_{m^d,\mu_C}$. All copulas with the property $C = \hat{C}$ were characterized in [14]. In particular, an associative copula C coincides with its survival copula $\hat{C}$ if and only if C is either a member of the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ or if C is a symmetric ordinal sum of Frank t-norms [12, 14].

Because of $T_0^{\mathbf{F}} = T_{\mathbf{M}}$, $T_1^{\mathbf{F}} = T_{\mathbf{P}}$, and $T_\infty^{\mathbf{F}} = T_{\mathbf{L}}$, for all Sugeno, Choquet and opposite Sugeno integrals we have (for X finite)

$$\Big(\int_X f\,dm\Big)^d = \int_X f\,dm^d.$$

4 Transformations of copulas

Here we present results from [16, 17]. If Φ denotes the set of all increasing bijections from $[0,1]$ to $[0,1]$, then for each $\varphi \in \Phi$ and for each $V \in \mathcal{V}$ consider the function $V_\varphi\colon [0,1]^2 \to [0,1]$ given by

$$V_\varphi(x,y) = \varphi^{-1}(V(\varphi(x), \varphi(y))).$$

It is obvious that $a \in [0,1]$ is an idempotent element of C if and only if $\varphi(a)$ is an idempotent element of C_φ, and M is the only copula which is φ-invariant for each $\varphi \in \Phi$. In general, the fact that C is a copula is neither necessary nor sufficient for C_φ being a copula. Now we first are interested under which conditions C_φ is a copula and under which conditions a copula C is φ-invariant.

4.1 Example
For $p \in \,]0,\infty[$ consider the function $\varphi_p \in \Phi$ defined by $\varphi_p(x) = x^p$.
(i) The product $T_{\mathbf{P}}$ is φ_p-invariant for each $p \in \,]0,\infty[$.
(ii) $T_{\mathbf{L}}$ is φ_p-invariant only if $p = 1$, and $T_{\mathbf{L}\varphi_p}$ is a copula only if $p \in \,]0,1]$.

The following result follows from [19, Theorem 7]:

4.2 Proposition
Assume that $V \in \mathcal{V}$ is associative and has neutral element 1, and let $\varphi \in \Phi$. Then V_φ is a copula if and only if for all $x, y, z \in [0,1]$

$$|\varphi^{-1}(V(x,z)) - \varphi^{-1}(V(y,z))| \leq |\varphi^{-1}(x) - \varphi^{-1}(y)|.$$

4.3 Theorem
For each $\varphi \in \Phi$ the following are equivalent:
(i) *The function φ is concave.*
(ii) *For each copula C the function C_φ is a copula.*

The class of increasing bijections φ_p introduced in Example 4.1 contains all transformations $\varphi_{1/n}$ mentioned in the introduction. Moreover, because of the Lipschitz continuity, a copula C is φ_p-invariant for each $p \in \,]0,\infty[$ if and only if C is $\varphi_{1/n}$-invariant for each $n \in \mathbb{N}$.

Following [8] (compare also [2]), a copula C^* is said to be the *maximum attractor of the copula C* (or, equivalently, *C belongs to the maximum domain of attraction of C^**) if for all $(x,y) \in [0,1]^2$ we have

$$\lim_{n\to\infty} C^n(x^{1/n}, y^{1/n}) = C^*(x,y).$$

It is obvious that each copula C which is φ_p-invariant for each $p \in]0, \infty[$ is a maximum attractor of itself, i.e., $C^* = C$. The set of all maximum attractor copulas will be denoted by $\mathcal{M}$. We have by [22, 26] (compare also [4]) that each maximum attractor copula C^* can be expressed in the form

$$C^*(x, y) = e^{\log(xy) \cdot A(\frac{\log x}{\log(xy)})}$$

for some A from the set

$$\{A \colon [0,1] \to [0,1] \mid A \text{ is convex and } \max(x, 1-x) \leq A(x) \text{ for all } x \in [0,1]\}.$$

It is obvious that $T_{\mathbf{P}}$ is the weakest maximum attractor and $T_{\mathbf{M}}$ is the strongest one. The class $\mathcal{M}$ is closed under suprema and weighted geometric means. Although $T_{\mathbf{L}}$ belongs to the maximum domain of attraction of $T_{\mathbf{P}}$, there are copulas not belonging to any maximum domain of attraction.

In the next proposition we give the relationship between φ_p-invariant copulas and the class $\mathcal{M}$ of maximum attractors.

4.4 Proposition
For a copula C, the following are equivalent:
(i) $C \in \mathcal{M}$.
(ii) $C_{\varphi_p} = C$ *for all* $p \in]0, \infty[$.
(iii) $C_{\varphi_p} = C_{\varphi_q} = C$ *for some* $p, q \in]0, \infty[$ *such that* $\frac{\log p}{\log q}$ *is irrational.*

5 Uniform approximation of associative copulas

We present here the result from [13]. The set $\mathcal{X} = [0,1]^{[0,1]^2}$ of all functions from the unit square $[0,1]^2$ into the unit interval $[0,1]$, will be equipped with the topology $\mathcal{T}_\infty$ induced by the metric $d_\infty \colon \mathcal{X}^2 \longrightarrow [0, \infty]$ given by $d_\infty(f, g) = \sup\{|f(x,y) - g(x,y)| \mid (x,y) \in [0,1]^2\}$ (corresponding to the uniform convergence). The class of associative copulas, i.e., of all 1-Lipschitz t-norms [12] is a compact subset of $\mathcal{X}$ (observe that this is not true for the class of all continuous t-norms).

The main result of [13] can be putted in the following theorem.

5.1 Theorem
The set $\mathcal{C}_{\mathbf{a}}$ of all associative copulas is the closure of both the set $\mathcal{C}_{\mathbf{s}}$ of all strict copulas and the set $\mathcal{C}_{\mathbf{ns}}$ of all non-strict Archimedean copulas.

As a consequence we have that each associative copula can be approximated with arbitrary precision by some strict as well as by some non-strict Archimedean copula. Notice that $\mathcal{C}_{\mathbf{s}}$ and $\mathcal{C}_{\mathbf{ns}}$ are disjoint sets whose union, i.e., the set of Archimedean copulas, is a proper subset of $\mathcal{C}_{\mathbf{a}}$. As it was proven in [12], the convergence of Archimedean copulas is strongly related to the convergence of their corresponding generators. To be precise, a sequence $(C_n)_{n\in\mathbb{N}}$ of Archimedean copulas with generators $(\varphi_n)_{n\in\mathbb{N}}$ converges to an Archimedean copula C

with generator φ if and only if there is a sequence of positive constants $(c_n)_{n\in\mathbb{N}}$ such that for each $x \in]0,1]$ we have $\lim_{n\to\infty} c_n \cdot \varphi_n(x) = \varphi(x)$.

Given two copulas C and D, consider their $*$-product $C * D$ introduced in (1) which is always a copula, i.e., the $*$-product is an operation on the set $\mathcal{C}$ of all copulas. Moreover, $(\mathcal{C}, *)$ is a non-commutative semigroup whose annihilator is the product $T_{\mathbf{P}}$ and whose neutral element is the minimum $T_{\mathbf{M}}$. As a consequence of Theorem 5.1 and [5, Theorem 2.3], for each associative copula C and for each copula D there are sequences of Archimedean and strict and non-strict Archimedean copulas $(C_n)_{n\in\mathbb{N}}$, respectively, such that the sequences $(C_n)_{n\in\mathbb{N}}$ and $(C_n * D)_{n\in\mathbb{N}}$ converge uniformly to C and $C * D$, respectively.

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