Vector Interpolative Logic

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Abstract: Vector interpolative logic (I-logic) is a consistent generalization of vector classical logic, so that the components of analyzed I-logic vectors have values from the real interval [0, 1]. All laws of classical logic and as a consequence, vector classical logic too, are preserved in the vector I- logic. This result is not possible in the frame of conventional fuzzy and/or MV- logic approaches.

Keywords: Logical Vector (LV); Primary, Atomic Combined LV; Structure of LV; Intensity (Value) of LV Component;

1 Introduction

In vector classical logic (vector logic) variables are logical vectors of the same dimension with only 0-1 components (components have classical logic values 1 or 0, true or false, respectively). Logical operations in vector case are logical operations on corresponding components of logic vectors. Vector logic, Section 2, is in the same frame as classical logic – Boolean algebra. Vector I-logic, Section 3, is a consistent generalization of vector logic, so that the components of analyzed vectors have values from the real interval [0, 1]. Vector I-logic has two levels: *Symbolic* (qualitative) and *Valued* (quantitative). On symbolic level the following notions are the basic: *Symbolic context* of vector I-logic is *a set of primary logic I-vectors*. *Algebra of I- logic vectors* (set of I-logic vectors generated by the set of primary I-vectors) is Boolean algebra. *Atomic I-logic vector* is the simplest element of Boolean algebra and non-atomic elements (combined I-logic vectors) are actually vector superposition of relevant atomic elements (atomic I-vectors). *Structure of an I- logic vector* is information about which atomic vectors are included and/or which are not included in it (relevant and/or not relevant for it). *Principle of structural functionality*: structure of any combined real logic vector can be determined directly on the base of the structures of its components. On valued level the following notions are the basic: *Valued context* of vector I-logic is vector universe – unit hyper cube of dimension corresponding to analyzed I-vectors (vector space of analyzed dimension, with components from real unit interval [0, 1]). *Intensity*, value of components of I-logic vector. *Intensity of (components) of atomic* I-logic vector is a function of the intensity of (components) of primary I-logic vectors and chosen *operator of generalized product*. Intensity of components of any I-logic vector is obtained as superposition of corresponding component intensities of relevant atomic I-vector. All laws of classical logic are preserved in the real vector logic. This result is not possible in the case of conventional fuzzy approaches.

2 Classical Vector Logic

In classical vector logic (vector logic) variables are of the same dimension with 0-1 components (logic vectors). Logical operations in vector case are logical operations on the corresponding elements of logic vectors.

Let $\vec{A}$ be a set of all logic vectors of the same dimension. A Boolean algebra of logic vectors is an algebraic structure $\left(\vec{A}, \wedge, \vee, \neg\right)$ with the following four additional properties:

1 *bounded below*: There exists an element (constant zero vector) $\vec{0}$, such that $\vec{a} \vee \vec{0} = \vec{a}$ for all $\vec{a}$ in $\vec{A}$.

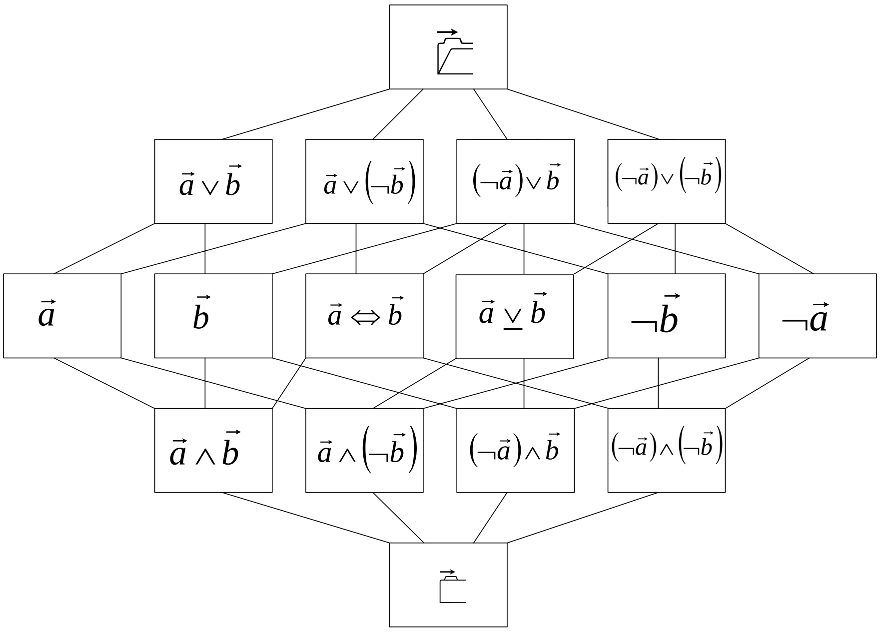
2 *bounded above*: There exists an element (constant unit vector) $\vec{1}$, such that $\vec{a} \wedge \vec{1} = \vec{a}$ for all $\vec{a}$ in $\vec{A}$.

3 *distributive law*: For all $\vec{a}, \vec{b}, \vec{c}$ in $\vec{A}$, $\vec{a} \wedge \left(\vec{b} \vee \vec{c}\right) = \left(\vec{a} \wedge \vec{b}\right) \vee \left(\vec{a} \wedge \vec{c}\right)$.

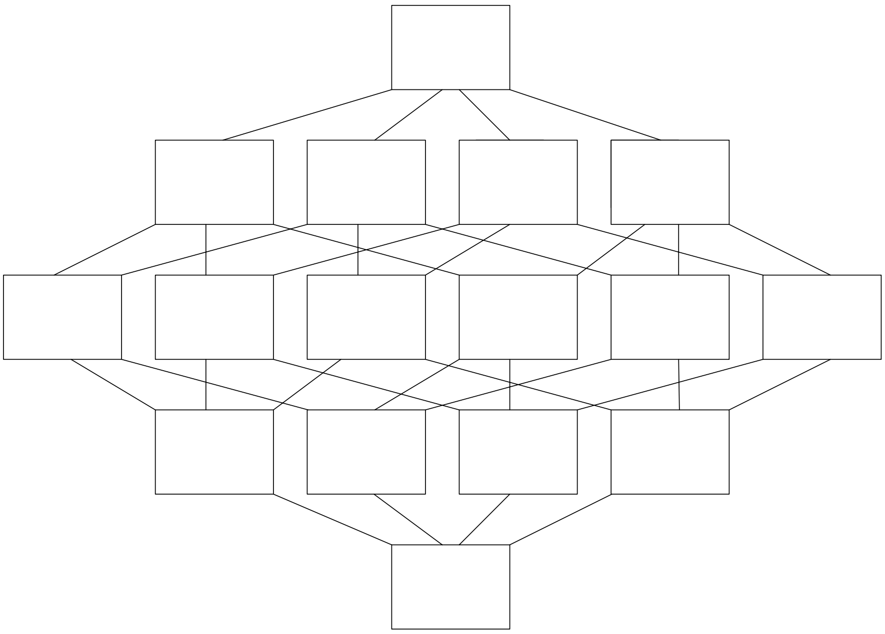
4 *existence of complements*: For every $\vec{a}$ in $\vec{A}$ there exists an element (complement vector) $\neg\vec{a}$ in $\vec{A}$ such that $\vec{a} \vee \neg\vec{a} = \vec{1}$ and $\vec{a} \wedge \neg\vec{a} = \vec{0}$.

Boolean lattice of logic vectors is illustrated by the following example:

Example: *Boolean lattice in the case of two primary logic vectors* $\Omega = \left\{\vec{a}, \vec{b}\right\}$



or in the following concrete case $\vec{a} =$ [illegible], $\vec{b} =$ [illegible]



3 Vector Interpolative Logic (I-logic)

Vector I-logic is a generalization of vector logic in the sense that vector component took values from real unit interval [0, 1], and not only from {0, 1}. Vector I-logic has two levels: (a) symbolic (qualitative) and (b) valued (quantitative).

3.1 Vector I-logic: Symbolic Level

On a symbolic level I-logic vectors are treated independently of their valued realization. So, a vector on a symbolic level in I-logic is valued irrelevant, which includes dimension of irrelevant ness too. The following notions are introduced and analyzed in I-logic on symbolic level: *primary*, *atomic* and *combined* I-logic vectors; *algebra* of I-logic vectors; *structure* of I-logic vector and *principle of structural functionality*.

3.1.1 Primary I-logic Vector

Primary I-logic vector is an element of symbolic (qualitative) context $\vec{\Omega}$, the finite I-logic vector set. Any primary logic vector can't be logical (Boolean) function of the remaining vectors from the symbolic context.

3.1.2 Atomic I-logic Vector

Atomic I-logical vector has the most simple structure (it doesn't include any other vector from algebra except itself and a trivial zero vector). To every elements of power set $P(\vec{\Omega})$ corresponds one atomic I-logic vectors, defined by the following expression:

$$\vec{\alpha}\left(\vec{A}\right) = \bigwedge_{\vec{a}_i \in \vec{A}} \vec{a}_i \bigwedge_{\vec{a}_j \in \vec{\Omega} \setminus \vec{A}} \neg \vec{a}_j$$

As a consequence if $n = \left|\vec{\Omega}\right|$ than the number, of the corresponding atomic I-logic vectors, is 2^n, since $\left|P\left(\vec{\Omega}\right)\right| =$ [illegible]^n.

Example: *Atomic I-logic vectors, in the case of two primary logical vectors* $\vec{\Omega} = \left\{\vec{a}, \vec{b}\right\}$*, are:*

$$\vec{\alpha}(\{a,b\}) = \vec{a} \wedge \vec{b}$$
$$\vec{\alpha}(\{a\}) = \vec{a} \wedge \neg\vec{b}$$
$$\vec{\alpha}(\{b\}) = \neg\vec{a} \wedge \vec{b}$$
$$\vec{\alpha}(\varnothing) = \neg\vec{a} \wedge \neg\vec{b}$$

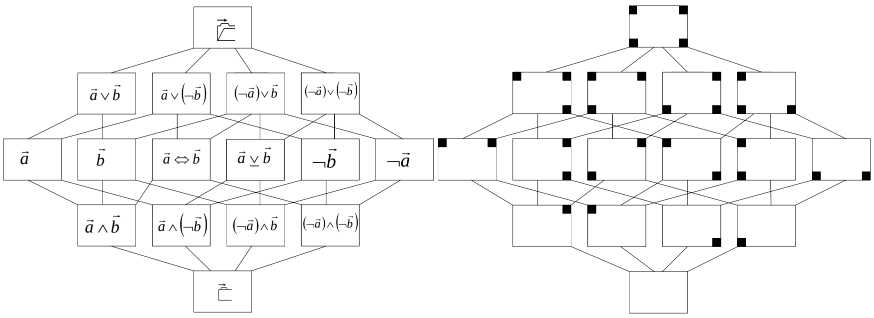
3.1.4 Structure of I-logic Vector

Structure of I-logic vector, on a symbolic level – symbolic structure, is symbolic context dependent. Structure is information about which atomic I-logic vectors are relevant for this logical vector and/or which are not (which are included and/or not included in it). Any I-logic vector is actually disjunction (joint) of *relevant* atomic I-logic vectors.

Note: *Structure of I-logic vector is context (set of primary vectors) dependent and it is not value-dependent. It means that a value realization including the dimension of analyzed vector, is irrelevant on the structure of I-logic vectors.*

Structures of logical vectors are illustrated on the following example:

Example: In the case $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ elements of lattice $\vec{B}(\vec{\Omega})$ as a function of atomic elements



are given in the following table:

Table

Structure of elements of lattice with 16 elements

	0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
	0	0	1	0	0	1	0	0	1	0	1	1	1	0	1	1
	0	0	0	1	0	0	1	0	1	1	0	1	0	1	1	1
	0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1

Illustration of structures of I-logic vectors is given in the following example.

***Example**: In the case $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ structures of elements of corresponding algebra, logic vector functions, is:*

s	$\{\vec{a},\vec{b}\}$	$\{\vec{a}\}$	$\{\vec{b}\}$	$\varnothing$
$\vec{a} \wedge \neg\vec{a}$	🗀	🗀	🗀	🗀
$\vec{a} \wedge \vec{b}$	🗁	🗀	🗀	🗀
$\vec{a} \wedge \neg\vec{b}$	🗀	🗁	🗀	🗀
$\neg\vec{a} \wedge \vec{b}$	🗀	🗀	🗁	🗀
$\neg\vec{a} \wedge \neg\vec{b}$	🗀	🗀	🗀	🗁
$\vec{a}$	🗁	🗁	🗀	🗀
$\vec{b}$	🗁	🗀	🗁	🗀
$\vec{a} \Leftrightarrow \vec{b}$	🗁	🗀	🗀	🗁

$(A) =$ the matrix above;

s	$\{\vec{a},\vec{b}\}$	$\{\vec{a}\}$	$\{\vec{b}\}$	$\varnothing$
$\vec{a} \vee \neg\vec{a}$	🗁	🗁	🗁	🗁
$\neg\vec{a} \vee \neg\vec{b}$	🗀	🗁	🗁	🗁
$\neg\vec{a} \vee \vec{b}$	🗁	🗀	🗁	🗁
$\vec{a} \vee \neg\vec{b}$	🗁	🗁	🗀	🗁
$\vec{a} \vee \vec{b}$	🗁	🗁	🗁	🗀
$\neg\vec{a}$	🗀	🗀	🗁	🗁
$\neg\vec{b}$	🗀	🗁	🗀	🗁
$\vec{a} \underline{\vee} \vec{b}$	🗀	🗁	🗁	🗀

$(A) =$ the matrix above.

3.1.4 Principle of Structural Functionality of I-logic Vectors

Principal of structural functionality, [1], of I-logic vectors: Structure of any I-logic vector can be calculated directly on the base of the structures of its components (corresponding I-logic vectors) and structures of logical connectives. Structures of logical connectives are given in the following way:

$(-)$	
🗀	🗁
🗁	🗀

$\cap$	🗀	🗁
🗀	🗀	🗀
🗁	🗀	🗁

$\cup$	🗀	🗁
🗀	🗀	🗁
🗁	🗁	🗁

Using this, each structure function s extends uniquely to a structure determination of all I-logic vectors as follows:

$$s(\vec{a} \wedge \vec{b})(\vec{S}) = s(\vec{a})(\vec{S}) \cap s(\vec{b})(\vec{S}),$$
$$s(\vec{a} \vee \vec{b})(\vec{S}) = s(\vec{a})(\vec{S}) \cup s(\vec{b})(\vec{S}),$$
$$s(\neg\vec{a})(\vec{S}) = (-)s(\vec{a})(\vec{S}),$$

$$\vec{S} \in \mathrm{P}(\vec{\Omega}), \quad \vec{a}, \vec{b} \in \vec{\mathrm{B}}(\vec{\Omega}).$$

This fundamental property has its isomorphism on the value level but only in classical case (values of logical variables are form {0, 1}), known as *principle of truth functionality*. Principle of truth functionality is not fundamental and as a consequence it can’t be used in generalization.

3.1.5 Combined Real Logic Vector

Combined logic vector is a logical function of primary logic vectors. Any logical vector can be represented by a canonical disjunctive form using atomic I-logical vectors and its structure. Canonical disjunctive form is actually disjunction of relevant atomic I-logic vectors.

3.1.6 Algebra of Real Vector Logic

Symbolic vector context $\vec{\Omega}$ is generator of the set of all I-logic vectors on symbolic level $\vec{B}(\vec{\Omega})$ – algebra of I-logic vectors. Algebra of I-logic vectors is Boolean algebra. A Boolean algebra of I-logic vectors is the following algebraic structure $\left(\vec{B}(\vec{\Omega}), \wedge, \vee, \neg\right)$ with the following four additional properties:

1. *bounded below*: There exists an element (constant zero vector) $\vec{0}$, such that $\vec{a} \vee \vec{0} = \vec{a}$ for all $\vec{a}$ in $\vec{B}(\vec{\Omega})$.
2. *bounded above*: There exists an element (constant unit vector) $\vec{1}$, such that $\vec{a} \wedge \vec{1} = \vec{a}$ for all $\vec{a}$ in $\vec{B}(\vec{\Omega})$.
3. *distributive law*: For all $\vec{a}, \vec{b}, \vec{c}$ in $\vec{B}(\vec{\Omega})$, $\vec{a} \wedge (\vec{b} \vee \vec{c}) = (\vec{a} \wedge \vec{b}) \vee (\vec{a} \wedge \vec{c})$.
4. *existence of complements*: For every $\vec{a}$ in $\vec{B}(\vec{\Omega})$ there exists an element (complement vector) $\neg\vec{a}$ in $\vec{B}(\vec{\Omega})$ such that $\vec{a} \vee \neg\vec{a} = \vec{1}$ and $\vec{a} \wedge \neg\vec{a} = \vec{0}$.

The same as in classical case as a consequence of the fact that symbolic level is valued irrelevant. In the case when the number of elements of context is $|\vec{\Omega}|$, the number of elements of algebra in general is $2^{2^{|\Omega|}}$.

3.2 Real Vector Logic: Valued Level

Valued context of vector I-logic is vector universe – unit hyper cube of dimension corresponding to analyzed I-vectors (vector space of analyzed dimension, with components from real unit interval [0, 1]). Results from symbolic level are treated on valued level as constraints and as a consequence they are preserved on valued level, contrary to fuzzy approaches. So, Boolean nature (all Boolean tautologies and/or contradictions) and/or Boolean lattice is preserved in a general case.

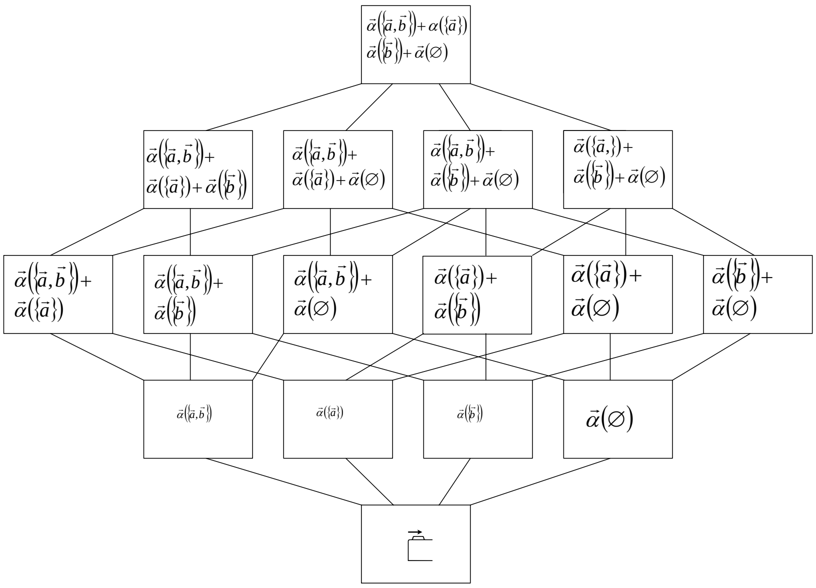
3.2.1 Superposition of Intensity of Atomic I-logic Vectors

Since conjunction of any two different atomic I-vectors is equal to constant zero vector, combined I-logic vector is actually superposition of relevant atomic I-logic vectors. This is illustrated in the following example:

Example*: In the case* $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ *combined I-logic vectors as superposition of relevant atomic I-logic vectors:*

$$\begin{bmatrix} \vec{a} \wedge \neg\vec{a} \\ \vec{a} \wedge \vec{b} \\ \vec{a} \wedge \neg\vec{b} \\ \neg\vec{a} \wedge \vec{b} \\ \neg\vec{a} \wedge \neg\vec{b} \\ \vec{a} \\ \vec{b} \\ \vec{a} \Leftrightarrow \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{a} \wedge \vec{b} \\ \vec{a} \wedge \neg\vec{b} \\ \neg\vec{a} \wedge \vec{b} \\ \neg\vec{a} \wedge \neg\vec{b} \end{bmatrix}, \qquad \begin{bmatrix} \vec{a} \vee \neg\vec{a} \\ \neg\vec{a} \vee \neg\vec{b} \\ \neg\vec{a} \vee \vec{b} \\ \vec{a} \vee \neg\vec{b} \\ \vec{a} \vee \vec{b} \\ \neg\vec{a} \\ \neg\vec{b} \\ \vec{a} \veebar \vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{a} \wedge \vec{b} \\ \vec{a} \wedge \neg\vec{b} \\ \neg\vec{a} \wedge \vec{b} \\ \neg\vec{a} \wedge \neg\vec{b} \end{bmatrix}.$$

Or in lattice representation



where:

$\vec{\alpha}(\{a,b\}) = \vec{a} \wedge \vec{b}, \quad \vec{\alpha}(\{a\}) = \vec{a} \wedge \neg\vec{b}, \quad \vec{\alpha}(\{b\}) = \neg\vec{a} \wedge \vec{b}$ *and* $\vec{\alpha}(\varnothing) = \neg\vec{a} \wedge \neg\vec{b}$.

3.2.2 Intensity of Atomic I-logic Vectors

Intensity of atomic I-logic vector is actually the intensity of its components. Intensities of components of atomic I-logic vectors are calculated in the way illustrated by the following example:

Example: *In the case* $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ *intensities of components of atomic I-logic vectors are:*

$$\alpha_i(\{\vec{a}, \vec{b}\}) = a_i \otimes b_i,$$
$$\alpha_i(\{\vec{a}\}) = a_i - a_i \otimes b_i,$$
$$\alpha_i(\{\vec{b}\}) = b_i - a_i \otimes b_i,$$
$$\alpha_i(\varnothing) = 1 - a_i - b_i + a_i \otimes b_i$$
$$i = 1, \ldots, n$$

3.2.3 Generalized Product

Operator $\otimes_{(n)}$ or abbreviated $\otimes$, is generalized n-product on real unit interval: $\otimes : [0,1]^n \to [0,1]$, such that for all $a_1, \ldots, a_n \in [0,1]$ the following five axioms are satisfied [2]:

(T1) *Commutativity*

$$\otimes(a_i, a_j) = \otimes(a_j, a_i),$$

(T2) *Associativity*

$$\otimes(a_i, \otimes(a_j, a_k)) = \otimes(\otimes(a_i, a_j), a_k),$$

(T3) *Monotonicicity*

$$\otimes(a_i, a_j) \le \otimes(a_i, a_k) \ \textit{whenever} \ a_j \le a_k,$$

(T4) *Boundary condition*

$$\otimes(a_i, 1) = a_i,$$

(T5) *Non-negativity condition*

$$\sum_{S \in P(\Omega \setminus A)} (-1)^{|S|} \underset{a_i \in A \cup S}{\otimes} a_i \ge 0 \quad \forall A \in P(\Omega)$$

where: $\Omega = \{a_1, \ldots, a_n\} \in [0,1]^n$.

Remark: Axioms (T1)-(T4) are the same as in the case of definition of T-norm, [3], a non-negativity condition is new. The role of operator of generalized product is only for interpolation (it is not logic (or relation) operator as it is the case with T-norm in fuzzy relations).

where: n is the dimension of analyzed I-logic vectors $\vec{a}$ *and* $\vec{b}$.

3.2.3 Intensity of I-logic Vectors

Intensity of I-logic vector is the intensity of its components. Intensities of the same component of all I-logic vectors generated by the same symbolic and valued contexts formed a Boolean lattice on valued level, as it is illustrated in the following example:

***Example**: Boolean lattice of intensities of the same component of all I-logic vectors generated by* $\vec{\Omega} = \{\vec{a}, \vec{b}\}$:

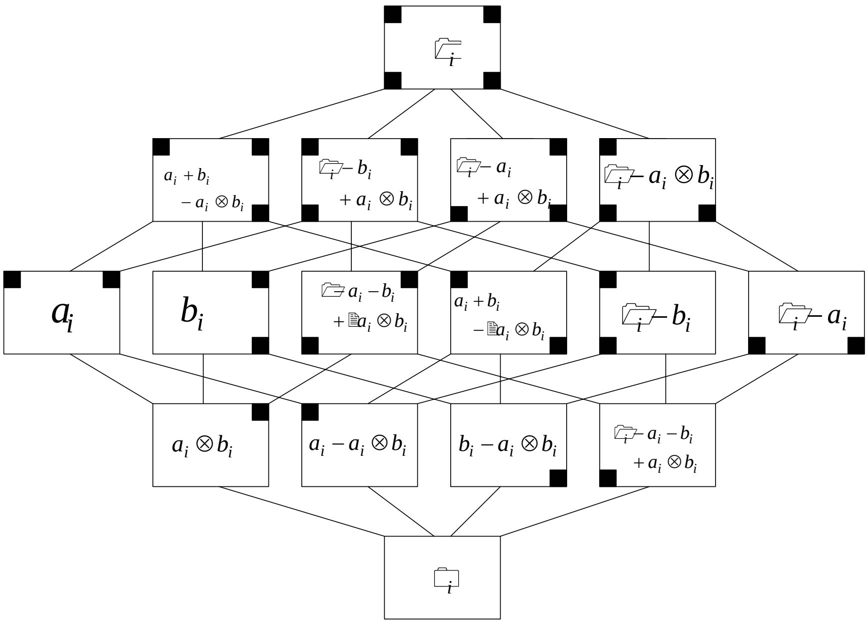
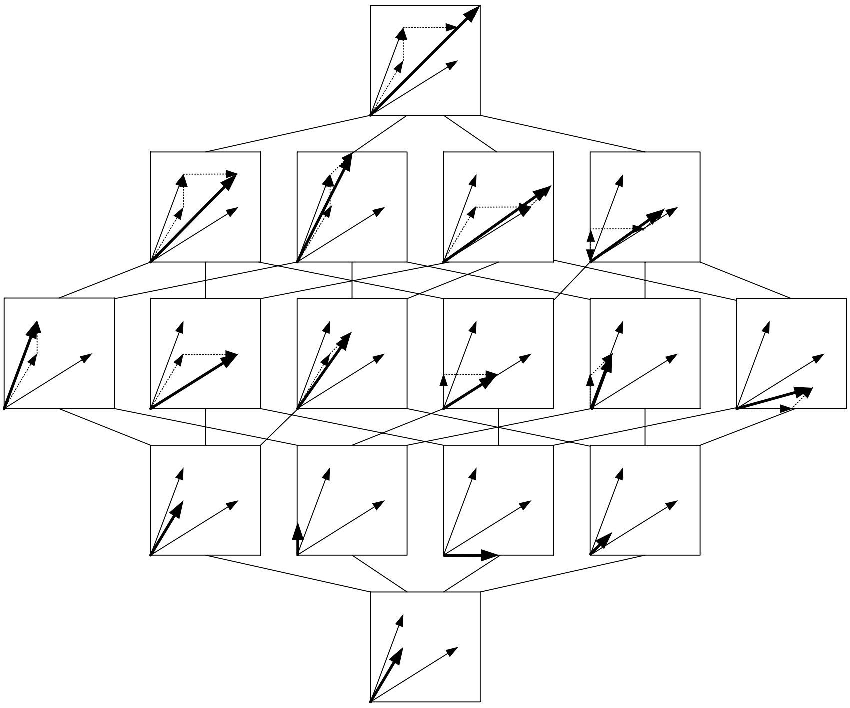


Figure
Boolean lattice of i-th component intensity functions of I-logic vectors

3.2.3 Boolean Lattice of I-logic Vectors: Example

Here is given an example of Boolean lattice of I-logic vectors.

***Example:** Boolean lattice generated by two primary I-logic vectors* $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ *dimension 2. Generalized product* $\otimes$ *in this example is **min** function:*



Figure

Boolean lattice of I-logic vectors generated by $\vec{\Omega} = \{\vec{a}, \vec{b}\}$ using *min* as generalized product $\otimes$

Conclusion

Vector interpolative logic (I-logic) is a generalized vector logic in the sense that the values of vector components are no more only from {0, 1}, but from the whole real unit interval [0, 1]. Vector I-logic is realized on the base of the interpolative logic. I-logic has two levels: (a) Symbolic (qualitative) and (b) Valued (quantitative). On a symbolic level vectors are treated as abstract notions, independently of their value of realization and even their dimension. Algebra of I-logic vectors on a symbolic level is Boolean algebra same as in classical case. A partial order (Boolean lattice) on a symbolic level is based on inclusion not on the value. A partial order on a valued level (Boolean lattice of values) is based on value and it is consistent with partial order on a symbolic level (Boolean lattice of inclusions). Applications of this theoretical result will be the subject of forthcoming papers.

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