Representation of the revised monotone functional by the Choquet integral with respect to signed fuzzy measure

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Abstract: The signed fuzzy measures are considered and some of their properties are shown. There is introduced the revised monotone functional and there are given conditions for its asymmetric Choquet integral-based representation.

Key words and phrases: revised monotonicity, signed fuzzy measure, Choquet integral.

1 Introduction

Due to its special non linear character, the Choquet integral with respect to a fuzzy measure, is one of the most popular and flexible aggregation operator [3, 5, 17]. The basic features of Choquet integral, defined for non-negative measurable functions, are monotonicity and comonotonic additivity, see [1, 3, 10]. For an exhaustive overview of applications of Choquet integral in the decision under uncertainty we recommend [2, 8, 9, 14, 15, 18]. Recall that non-negative set function m such that $m(\emptyset) = 0$ and $A \subset B$ implies $m(A) \leq m(B)$ (monotonicity), is called by various names, such as *capacity, non-additive measure, fuzzy measure.*

A generalized fuzzy measure, *a signed fuzzy measure*, introduced by Liu in [6], is a revised monotone, real-valued set function, vanishing at the empty set, see [10]. Murofushi et al. in [7] used term *non-monotonic fuzzy measure* to denote a real-valued set function satisfying $m(\emptyset) = 0$. In this paper we deal with a signed fuzzy measure in the sense of definition given in [6].

The properties of two usual extensions of Choquet integral to the class of all measurable functions have been studied by various authors [1, 3, 7, 10]. The first one extension, *the symmetric* Choquet integral, introduced by Šipoš, is homogeneous with respect to multiplication by a real constant and the second one, *the asymmetric* Choquet integral is comonotone additive and homogeneous with respect to multiplication by a non-negative constant. In both cases the monotonicity is violated. The asymmetric Choquet integral is defined with respect to a real-valued set function m, not necessary monotone.

The fuzzy integral defined with the use of maximum and minimum operators was introduced by Sugeno in [16]. The Sugeno integral is defined on the class of functions whose range is contained in $[0,1]$ and with respect to a normalized fuzzy measure. It is comonotone-$\vee$-additive (comonotone maxitive), $\wedge$-homogeneous and monotone functional. An extension of the Sugeno integral in the spirit of the symmetric extension of Choquet integral is proposed by M. Grabisch in [4]. The symmetric Sugeno integral is neither monotone nor commonotone -$\vee$-additive in general. In the paper [13] authors considered a representation by two Sugeno integrals of the functional L defined on the class of functions $f : X \to [-1,1]$ on a finite set X. In the case of infinitely countable set X there was obtained that the symmetric Sugeno integral is comonotone-$\circledvee$-additive functional on the class of functions with finite support.

In this contribution we will deal with a revised monotonicity of a real-valued set function m, $m(\emptyset) = 0$ and asymmetric Choquet integral with respect to m. In the next section the short overview of basic notions and definitions is given. In Section 3 we consider a revised monotonicity of real-valued set functions vanishing at the empty set. Finally, in Section 4 we introduce a revised monotone functional and discuss the conditions for its asymmetric Choquet integral-based representation.

2 Preliminaries

Let $X = \{x_1, \ldots, x_n\}$ be a finite set. Let $\mathcal{P}(X)$ be class of subsets of universal set X. We have by [6, 10] the following definition.

Definition 1 *A real-valued set function $m : \mathcal{P}(X) \to \mathbb{R}$, is a signed fuzzy measure if it satisfies*
(i) $m(\emptyset) = 0$
(ii) (RM) *If $E, F \in \mathcal{P}(X)$, $E \cap F = \emptyset$, then*

$$\begin{aligned} &a)\ m(E) \geq 0,\ m(F) \geq 0,\ m(E) \vee m(F) > 0 &&\Rightarrow\quad m(E \cup F) \geq m(E) \vee m(F);\\ &b)\ m(E) \leq 0,\ m(F) \leq 0,\ m(E) \wedge m(F) < 0 &&\Rightarrow\quad m(E \cup F) \leq m(E) \wedge m(F);\\ &c)\ m(E) > 0,\ m(F) < 0 &&\Rightarrow\quad m(F) \leq m(E \cup F) \leq m(E). \end{aligned}$$

The conjugate set function of real-valued set function m, $m : \mathcal{P}(X) \to \mathbb{R}$ is defined by $\bar{m}(E) = m(X) - m(\bar{E})$, where $\bar{E}$ denotes the complement set of E, $\bar{E} = X \setminus E$. Obviously, if m is fuzzy measure, $\bar{m}$ is fuzzy measure, too.

However, if m is a signed fuzzy measure, its conjugate set function $\bar{m}$ need not to be a signed fuzzy measure and this fact will be discussed in the next section.

In the next example, introduced in [12], there is introduced a signed fuzzy measure m and we give an interpretation of the condition (RM) of the revised monotonicity of m in an application.

Example 1 *Let X be a set of $2n$ elements. Let $A, B \subset X$ such that $X = A \cup B$, $A \cap B = \emptyset$ and $card(A) = card(B) = n$. We define the set function $m : \mathcal{P}(X) \to \mathbb{R}$ by:*

$$m(E) = \begin{cases} card\ (X), & E = A \\ -card\ (X), & E = B \\ card\ (A \cap E) - card\ (B \cap E), & else. \end{cases}$$

m is a signed fuzzy measure.

We discuss the revised monotonicity of m. Same as in the modified version of the example a workshop, given by Murofushi et al. in [7], let us consider the set X as the set of all workers in a workshop, and sets A and B are the sets of good and bad workers in sense of their efficiency, i.e., inefficiency. If we suppose that workers from group A work two times better if they work all together (with nobody else), and workers from B two times worse, and in the other cases "anybody is effective in the proportion to its quantitative membership to the 'good' group A or 'bad' group B". The set function m is used to denote the efficiency of the worker. The interpretation of revised monotonicity is in the assumptions that for disjoint groups E of 'good' and F of 'bad' workers, if they work together, then their productivity is not greater to productivity of E and not less to productivity of F, for groups E and F of 'good'('bad') workers the simultaneous productivity is not less (not greater) to theirs individual productivity. Also, we have $m(X) = 0$, i.e., the productivity of all workers in the workshop equals to zero.

Let f be a real-valued function on X. We denote $f(x_i) = f_i$ for $i = 1, 2, \ldots n$ and $\mathcal{F}$ denotes class of all real-valued functions on X. *The asymmetric Choquet integral* with respect to a set function $m : \mathcal{P}(X) \to \mathbb{R}$ of function $f : X \to \mathbb{R}$ is given by

$$C_m(f) = \sum_{i=1}^{n} (f_{\alpha(i)} - f_{\alpha(i-1)}) m(E_{\alpha(i)}),$$

where f admits a comonotone-additive representation $f = \sum_{i=1}^{n} f_{\alpha(i)} \mathbf{1}_{E_{\alpha(i)}}$ and $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$ is a permutation of index set $\{1, 2, \ldots, n\}$ such that

$$f_{\alpha(1)} \leq f_{\alpha(2)} \leq \cdots \leq f_{\alpha(n)},$$

$f_{\alpha(0)} = 0$, sets $E_{\alpha(i)}$ are given by $E_{\alpha(i)} = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$ and $\mathbf{1}_E$ is characteristic function of set E, $E \subset X$. The asymmetric Choquet integral can be expressed in the terms of the Choquet integrals of non-negative functions f^+

and f^-, the positive and negative part of function f, i.e.

$$C_m(f) = C_m(f^+) - C_{\bar{m}}(f^-), \tag{1}$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$, and $\bar{m}$ is the conjugate set function of m.
Recall that two functions f and g on X are called *comonotone* [3] if for all $x\,,\,x_1 \in X$ we have $f(x) < f(x_1) \;\;\Rightarrow\;\; g(x) \le g(x_1)$. The asymmetric Choquet integral is a *comononotone additive* functional on $\mathcal{F}$, i.e. for all comonotone functions $f, g \in \mathcal{F}$ we have

$$C_m(f+g) = C_m(f) + C_m(g).$$

3 Signed fuzzy measure

In this section we will consider a signed fuzzy measure m with $m(X) = 0$. We will examine when its conjugate set function $\bar{m}$ is a signed fuzzy measure, too. Note that for a non-negative (non-positive) signed fuzzy measure m, condition $m(X) = 0$ implies $m(E) = 0$ for all $E \in P(X)$. In the sequel we suppose that $m : \mathcal{P}(X) \to \mathbb{R}$ is a signed fuzzy measure of non-constant sign. We easily obtain the next lemma by definition of signed fuzzy measure and the condition $m(X) = 0$.

Lemma 1 *Let m be a signed fuzzy measure, $m(X) = 0$. $m(E)$ and $m(\bar{E})$ are the opposite sign values, i.e.,*

$$(\forall E \in \mathcal{P}(X))\;\; (m(E) > 0 \;\Leftrightarrow\; m(\bar{E}) < 0).$$

Definition 2 *We say that a real-valued set function m, $m(\emptyset) = 0$ satisfies an intersection property if for all $E\,,F \in \mathcal{P}(X)$, $E \cap F \neq \emptyset$ and $E \cup F = X$ we have*

$$\begin{aligned}
&a)\; m(E) \ge 0,\, m(F) \ge 0,\, m(E) \vee m(F) > 0 &&\Rightarrow && m(E \cap F) \ge m(E) \vee m(F);\\
&b)\; m(E) \le 0,\, m(F) \le 0,\, m(E) \wedge m(F) < 0 &&\Rightarrow && m(E \cap F) \le m(E) \wedge m(F);\\
&c)\; m(E) > 0,\, m(F) < 0 &&\Rightarrow && m(F) \le m(E \cap F) \le m(E).
\end{aligned}$$

We have the next theorem.

Theorem 1 *Let m be signed fuzzy measure such that $m(X) = 0$. m has an intersection property if and only if the conjugate set function $\bar{m}$ of m is a signed fuzzy measure.*

`Proof.` Let m be a signed fuzzy measure with $m(X) = 0$.
$(\Longrightarrow)$ First, we suppose that m has an intersection property. We will prove that $\bar{m}$ is a signed fuzzy measure.
(i) Directly by definition of $\bar{m}$ we have $\bar{m}(\emptyset) = 0$.

(ii) In order to prove condition (RM) $a)$ let $E, F \in \mathcal{P}(X)$ such that $E \cap F = \emptyset$ and $\bar{m}(E) \geq 0$, $\bar{m}(F) \geq 0$, $\bar{m}(E) \vee \bar{m}(F) > 0$. We have $\bar{E} \cup \bar{F} = X$ and $m(\bar{E}) \leq 0, \;\; m(\bar{F}) \leq 0$ and $m(\bar{E}) \wedge m(\bar{F}) < 0.$ *(a)*
If we suppose that $\bar{E} \cap \bar{F} = \emptyset$ then we have $F = \bar{E}$. By Lemma 1. we obtain that the values $m(F)$ and $m(\bar{F})$ are the opposite sign values and it is in contradiction with *(a)*. Therefore, $\bar{E} \cap \bar{F} \neq \emptyset$. By the intersection property of m we have:

$$\begin{aligned} m(\bar{E} \cap \bar{F}) \leq m(\bar{E}) \wedge m(\bar{F}) &\iff m(\overline{E \cup F}) \leq m(\bar{E}) \wedge m(\bar{F}) \\ &\iff -\bar{m}(E \cup F) \leq (-\bar{m}(E)) \wedge (-\bar{m}(F)) \\ &\iff \bar{m}(E \cup F) \geq \bar{m}(E) \vee \bar{m}(F). \end{aligned}$$

Hence, we have that $\bar{m}$ satisfies condition (RM) a). Similarly we obtain that $\bar{m}$ satisfies conditions (RM) b) and c), hence, $\bar{m}$ is a signed fuzzy measure.
($\Longleftarrow$) Let $\bar{m}$ be a signed fuzzy measure, i.e. $\bar{m}$ is a revised monotone set function and $\bar{m}(\emptyset) = 0$. We obtain the claim directly by definition of the intersection property and the above consideration. □

Example 2 *Let m be a set function defined at the Example 1. m is a signed fuzzy measure with $m(X) = 0$. Obviously, m has an intersection property. Its conjugate set function $\bar{m} : \mathcal{P}(X) \to \mathbb{R}$ is defined by:*

$$\bar{m}(E) = \begin{cases} card\ (X), & E = A \\ -card\ (X), & E = B \\ card\ (B \setminus E) - card\ (A \setminus E), & else. \end{cases}$$

$\bar{m}$ is a signed fuzzy measure. Moreover, we have $m = \bar{m}$.

4 Revised monotone functional

In this section we focus on the asymmetric Choquet integral with respect to a signed fuzzy measure. As it is mentioned before, the monotonicity is violated. We will discuss the modification of monotonicity property, the revised monotonicity of asymmetric Choquet integral.

A real valued functional L, $L : \mathcal{F} \to \mathbb{R}$, defined on the class of functions $f : X \to \mathbb{R}$, can be viewed as an extension of a signed fuzzy measure m, so it is reasonable to require that $L(\mathbf{1}_E) = m(E)$, for all $E \in \mathcal{A}$ ($\mathbf{1}_E$ denotes characteristic function of set $E \subset X$). In order to examine the properties of a real valued functional L, under which it can be represented by the asymmetric Choquet integral w.r.t. a signed fuzzy measure, it is useful to consider the concept of comonotone functions.

The functional L is *comonotone additive* iff

$$L(f + g) \;\; = \;\; L(f) + L(g)$$

for all comonotone functions $f,\, g \in \mathcal{F}$. We say that functional L is *positive homogeneous* iff

$$L(a\,f) \quad = \quad a\,L(f)$$

for all $f \in \mathcal{F}$ and $a \geq 0$.

We introduce a revised monotone functional L defined on $\mathcal{F}$, see [12].

Definition 3 *Let $L : \mathcal{F} \to \mathbb{R}$ be a functional on $\mathcal{F}$.*
(i) L is revised monotone if and only if

$$\begin{array}{llll} a) & L(f) \geq 0,\, L(g) \geq 0,\, L(f) \vee L(g) > 0 & \Rightarrow & L(f+g) \geq L(f) \vee L(g) \\ b) & L(f) \leq 0,\, L(g) \leq 0,\, L(f) \wedge L(g) < 0 & \Rightarrow & L(f+g) \leq L(f) \wedge L(g) \\ c) & L(f) > 0,\, L(g) < 0 & \Rightarrow & L(g) \leq L(f+g) \leq L(f) \end{array}$$

for all functions $f,\, g \in \mathcal{F}$.
(ii) L is comonotone revised monotone if and only if conditions a), b) and c) are satisfied for all comonotone functions $f,\, g \in \mathcal{F}$.

Note that for a non-negative functional L acting on non-negative functions on X, the revised monotonicity ensures the monotonicity.

Directly by definitions of the comonotone additive and the revised monotone functional L we have the next proposition.

Proposition 1 *The asymmetric Choquet integral w.r.t. a signed fuzzy measure m , $C_m : \mathcal{F} \to \mathbb{R}$ is a comonotone revised monotone functional.*

Remark 1 Note that any additive functional $L : \mathcal{F} \to \mathbb{R}$ is a revised monotone functional. The Lebesgue integral with respect to a signed measure μ is a revised monotone functional.

We have the next theorem.

Theorem 2 *Let L be a real valued, revised monotone, positive homogeneoues and comonotone additive functional on $\mathcal{F}$. Then there exists a signed fuzzy measure m_L, such that L can be represented by the asymmetric Choquet integral w.r.t. m_L, i.e.,*

$$L(f) = C_{m_L}(f).$$

`Proof.` Let m be a set function m defined by

$$m_L(E) = L(\mathbf{1}_E), \text{ for } E \subseteq X.$$

Observe that for comonotone functions $\mathbf{1}_X$ and $-\mathbf{1}_E$, we have

$$m_L(\bar{E}) = L(\mathbf{1}_{\bar{E}}) = L(\mathbf{1}_X + (-\mathbf{1}_E)) = L(\mathbf{1}_X) + L(-\mathbf{1}_E) = m_L(X) + L(-\mathbf{1}_E),$$

hence

$$L(-\mathbf{1}_E) = -\bar{m}_L(E),\ E \subseteq X.$$

By definition of m_L and revised monotonicity of functional L we have:
1) $m_L(\emptyset) = L(\mathbf{1}_\emptyset) = L(0) = 0$
2) a) for $E\,, F \in \mathcal{A}$, $E \cap F = \emptyset$, and
$m_L(E) \geq 0$, $m_L(F) \geq 0$, $m_L(E) \vee m_L(F) > 0$ we have

$$\begin{aligned} m_L(E \cup F) &= L(\mathbf{1}_{E\cup F}) = L(\mathbf{1}_E + \mathbf{1}_F) \\ &\geq L(\mathbf{1}_E) \vee L(\mathbf{1}_E) = m_L(E) \vee m_L(F). \end{aligned}$$

Analogously, we obtain that m_L satisfies conditions (RM) b) and c), hence m_L is the revised monotone set function, so it is a signed fuzzy measure. Now, we consider $f \in \mathcal{F}$ and its comonotone additive representation $f = f^+ + (-f^-)$, where

$$\begin{aligned} f^+ &= \sum_{i=1}^{n} (a_i - a_{i-1}) \mathbf{1}_{E_i}\,, \\ -f^- &= \sum_{i=1}^{n} (b_i - b_{i+1})(-\mathbf{1}_{F_i})\,, \end{aligned}$$

$a_i = f^+_{\alpha(i)}$, $a_0 = 0$, $b_i = f^-_{\alpha(n+1-i)}$, $b_{n+1} = 0$,
a_i's are in non-decreasing, b_i's are in non-increasing order, α is a permutation, such that $-\infty < f_{\alpha(1)} \leq \cdots \leq f_{\alpha(n)} < \infty$, $E_i = E_{\alpha(i)}$,
$F_i = E_1 \setminus E_{\alpha(n+2-i)}$, $E_{\alpha(i)} = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$ and $E_{\alpha(n+1)} = \emptyset$.
For every i and j the functions $\mathbf{1}_{E_i}$ and $\mathbf{1}_{E_j}$ are comonotone, and by comonotone additivity and positive homogeneity of the functional L, we have

$$\begin{aligned} L(f^+) &= \sum_{i=1}^{n} (a_i - a_{i-1}) L(\mathbf{1}_{E_i}) \\ &= \sum_{i=1}^{n} (a_i - a_{i-1}) m_L(E_i) \\ &= C_{m_L}(f^+) \end{aligned}$$

and

$$\begin{aligned} L(-f^-) &= \sum_{i=1}^{n} (b_i - b_{i+1}) L(-\mathbf{1}_{F_i}) \\ &= -\sum_{i=1}^{n} (b_i - b_{i+1})(-L(-\mathbf{1}_{F_i})) \\ &= -\sum_{i=1}^{n} (b_i - b_{i+1}) \bar{m}_L(F_i) \\ &= -C_{\bar{m}_L}(f^-). \end{aligned}$$

Therefore by the comonotonicity of functions f^+ and $-f^-$ we obtain that

$$\begin{aligned} L(f) &= L(f^+ + (-f^-)) \\ &= L(f^+) + L(-f^-) \\ &= C_{m_L}(f^+) - C_{\bar{m}_L}(f^-) \\ &= C_{m_L}(f). \end{aligned}$$

□

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