On Morse-Smale complexes and dual subdivisions

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Abstract: The aim of this paper is to underline the links between some concepts and methodologies common to different research areas. In particular, ascending and descending Morse complexes, which describe a decomposition of the domain $\mathbb{M}$ *of a scalar function* f*, are related to a pair of dual subdivisions* S *and* S^* *of* $\mathbb{M}$*. A Morse-Smale complex can be viewed as an overlay of* S *and* S^**, and, as a consequence, (anti)cancellation of critical points of* f *can be interpreted as an Euler operation on* S*. Thus, by shifting the attention from regions, covered by cells involved in modifications, to edges and operations on them, it follows that any feasible (anti)cancellation preserves the structure of the Morse-Smale complex, and that the only restriction on the application of such operation is induced by the values of* f *at the vertices involved in an (anti)cancellation.*

Key words and phrases: Morse and Morse-Smale complexes, subdivisions and dual subdivisions, cancellation, anticancellation, Euler operations, 2D scalar fields, morphological representation.

1 Introduction

We want to point out some similarities between results in different research areas, and to model structures defined in one research field using structures defined in another field. The structures we want to compare are a Morse-Smale complex, which is introduced in scientific visualization for modeling the topology of a scalar function f defined over a manifold $\mathbb{M}$, and a pair of dual subdivisions of $\mathbb{M}$, introduced in computational geometry primarily for modeling Voronoi diagram and Delaunay triangulation. We will show that these structures are basically the same, apart from the fact that a Morse-Smale complex MS arises from the study of a scalar function f, and thus each vertex p in MS has an associated function value $f(p)$. Moreover, the operations for incremental modification of these structures are the same, and can be expressed one through the other. This insight enables us to remove some limitations in the application of a feasible (anti)cancellation of critical points in a Morse-Smale complex.

The remainder of this paper is organized as follows. In Section 2, we introduce some background notions. In Section 3, we introduce and compare Morse-Smale complexes and dual subdivisions. In Section 4, we describe and compare elementary simplification and refinement operations of structures introduced in Section 3. Section 5 concludes the paper.

2 Background Notions

In this Section, we present some background notions on cell complexes and Morse theory, that we will use in the rest of the paper.

2.1 Cell Complexes

Intuitively, a cell complex is a collection of basic elements, called cells, which cover a domain in the Euclidean space $\mathbb{E}^n$ [3]. For more information on algebraic topology, see [4].

A *k-dimensional cell* (k-cell) in $\mathbb{E}^n$, $0 \leq k \leq n$, is a subset of $\mathbb{E}^n$ homeomorphic to an open k-dimensional disk $B^k = \{x \in \mathbb{E}^k : ||x|| < 1\}$. A 0-cell is a point in $\mathbb{E}^n$. The *relative boundary* $b(\gamma)$ of a k-cell γ, $1 \leq k \leq n$, is the boundary of γ with respect to the topology induced by the usual topology of $\mathbb{E}^n$. The relative boundary of a 0-cell is empty. A *cell complex* Γ is a finite set of cells in $\mathbb{E}^n$, such that the cells are disjoint, and the boundary of each k-dimensional cell is the union of cells of Γ of dimension less than k. The *combinatorial boundary* $B(\gamma)$ of a cell γ is the collection of cells γ' of Γ such that $\gamma' \subseteq b(\gamma)$ (as a point set). A cell γ' on the boundary of a cell γ is called a *face* of γ. The maximum d of dimensions of cells γ over all cells of a complex Γ is called the *dimension* or the *order* of Γ. A d-dimensional complex is called *regular* if each k-cell γ', $0 \leq k \leq d$, is a face of some d-cell γ. The *domain* (or *carrier*) $\Delta\Gamma$ of a cell complex Γ is the subset of $\mathbb{E}^n$ spanned by the cells of Γ. We are interested in 2-dimensional regular cell complexes with manifold domain. In Figure 1 (left) such a complex is illustrated, whose domain is the extended plane (Euclidean plane compactified by the addition of a point at infinity), or, equivalently, a sphere.

2.2 Morse Theory

In short, Morse theory is a study of relationships between the topological shape of a manifold, and (the critical points of) a function defined on a manifold. We review here the basic notions of Morse theory in the case of 2-manifolds. For more details on Morse theory, see [6, 5].

We consider a C^2-differentiable real-valued function f (also called scalar field, or elevation function) defined over a compact 2-dimensional manifold $\mathbb{M}$. A point $p \in \mathbb{M}$ is a *critical point* of f if and only if the gradient ∇f of f vanishes on p ($\nabla f(p) = 0$). Function f is said to be a *Morse function* when

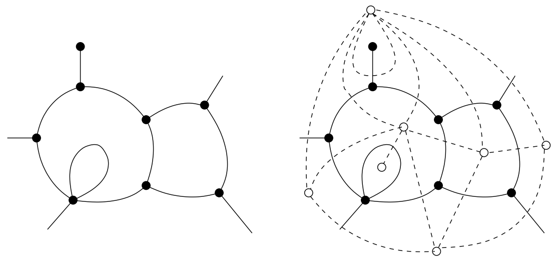

Figure 1: (left) A subdivision S of the extended plane. It has the structure of a regular 2-dimensional cell complex. (right) A quadrangulation of the extended plane obtained as an overlay of a subdivision S (bold) and its dual S^* (dashed).

all its critical points are non-degenerate, i.e., when the Hessian matrix $Hess_p f$ of the second order derivatives of f at p is non-singular (its determinant is $\neq 0$). This implies that the critical points of f are isolated. Assuming a local coordinate system, in which $f = f(x_1, x_2)$, we have that $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})$ and $Hess_p f = [\frac{\partial^2 f}{\partial x_i \partial x_j}(p)]^2_{i,j=1}$ The number of negative eigenvalues of $Hess_p f$ is called the *index* of a critical point p. In 2D, there are three types of non-degenerate critical points. A critical point p is a *minimum*, a *saddle*, or a *maximum* when p has index 0, 1 or 2, respectively. An *integral line* of a function f is a maximal path which is everywhere tangent to the gradient vector field ∇f of f. The classical Taylor formula shows that integral lines follow the gradient directions in which the function has the maximum increasing growth. Integral lines cannot be closed, nor infinite, and they cover $\mathbb{M}$. An integral line which connects a minimum to a saddle, or a saddle to a maximum is called a *separatrix*. From each saddle s there are two separatrices which connect s to maxima, and two separatrices which connect s to minima.

3 Partitions of a Manifold

In this section, we review and relate two ways to partition a manifold $\mathbb{M}$ into a cell complex. One is a general subdivision S (and an overlay of S and its dual S^*), and the other is a Morse complex (and a Morse-Smale complex). These two partitions have the same structural properties, but a Morse-Smale complex has an additional information content, related to elevation values.

3.1 Morse and Morse-Smale complexes

Let $f : \mathbb{M} \to \mathbb{R}$ be a Morse function, where $\mathbb{M}$ is a compact 2-dimensional manifold. Integral lines that converge to (originate from) a critical point p of index i form an i-cell ((2-i)-cell) called a *stable* or *descending (unstable* or *ascending) manifold* of p. The ascending (descending) manifolds are pairwise disjoint and

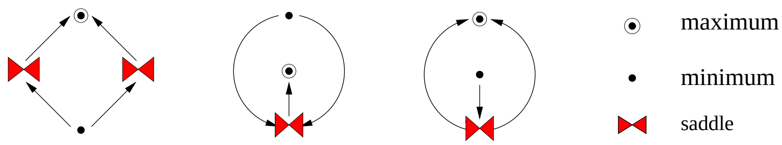

Figure 2: A generic 2-cell of a Morse-Smale complex (left). Two types of 2-cells glued along the boundary: an isolated mountain (middle), and a crater (right).

decompose $\mathbb{M}$ into open cells which form a complex, since the boundary of every cell is the union of lower-dimensional cells. Such complexes are called *ascending (unstable)* and *descending (stable) Morse complexes*. A Morse function $f : \mathbb{M} \rightarrow \mathbb{R}$ is called a *Morse-Smale function* if the ascending and descending manifolds intersect transversally. This means that such manifolds cross when they intersect, and that the crossing point is a saddle. Cells that are obtained as the intersection of the ascending and descending complex of a Morse-Smale function f decompose $\mathbb{M}$ into a *Morse-Smale complex*. Cells of dimension 0, 1 and 2 in this complex are called *vertices, edges* and *regions*, respectively. Each saddle s is incident to four separatrix lines, two ascending (connecting s to maxima) and two descending (connecting s to minima), which alternate around s. Each region of a Morse-Smale complex is a quadrangle (2-cell) whose vertices are critical points of f of index 0,1,2,1 (minimum, saddle, maximum, saddle), in this order along the boundary. For a Morse-Smale function f, there are three possible types of regions, also called slope districts in [7], and they are illustrated in Figure 2. The first one is a generic quadrangle, in which all the four vertices are distinct, Figure 2 (left), and the other two, glued along the boundary, correspond to an isolated mountain or a crater, Figure 2 (middle) and (right), respectively. Each 2-cell of a Morse-Smale complex consists of integral lines which originate from the same minimum, and converge to the same maximum.

3.2 Subdivisions, Dual Subdivisions

We will consider the case of a compact orientable 2-manifold $\mathbb{M}$ with a fixed orientation. For subdivisions of general compact 2-manifolds, see [2].

A *subdivision* S of $\mathbb{M}$ is a partition of $\mathbb{M}$ into a finite number of open elements: vertices (0-cells), edges (1-cells), and faces (2-cells), such that the boundary of every face is a closed path of (not necessarily distinct) edges and vertices. The closure of a face need not be homeomorphic to a closed 2-disk, but every vertex and every edge must be incident to some face. Thus, S is a regular 2-dimensional cell complex with domain $\mathbb{M}$. An example of a subdivision of the extended plane is illustrated in Figure 1 (left).

Let us now consider a subdivision S in which every edge e is directed. This is necessary if we want to distinguish the two faces incident at e as left and right face, and the two vertices incident at e as initial and terminal vertex.

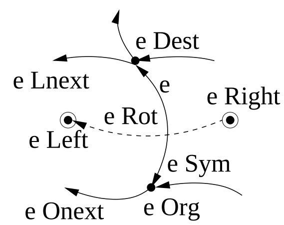

Figure 3: An edge e in a subdivision, with basic edge functions.

Let $eSym$ be the same edge as e, but with opposite direction, let $eOrg$ and $eDest$ be the initial and the terminal vertex of e, and let $eLeft$ and $eRight$ be the two adjacent faces on the left and right side of e. With $eLnext$ and $eOnext$ we denote the next counterclockwise edge e_1 around $eLeft$ such that $e_1Left = eLeft$, and the next counterclockwise edge e_2 around $eOrg$ such that $e_2Org = eOrg$. These notions are illustrated in Figure 3.

Two subdivisions S and S^* are *dual* to each other if for each edge e of either subdivision there is an edge $eRot$ of the other such that (i) $eRot^2 = eSym$, (ii) $(eSym)Rot = (eRot)Sym$, and (iii) $eLnext = eRot^{-1}OnextRot$. This correspondence of edges of S and S^* establishes a correspondence between (primal) faces and vertices of S and (dual) vertices and faces of S^*, respectively. Two vertices of one subdivision are connected by an edge if the corresponding faces of the other are adjacent along an edge. When each vertex of one subdivision is in the corresponding face of the other, and an edge crosses only its dual, then S and S^* are *strict duals*. For a subdivision S illustrated in Figure 1 (left), its dual S^* (together with S) is illustrated in Figure 1 (right).

If S and S^* are overlayed by adding one (skew) vertex v_e on each edge e of S and one (dual) vertex v_f in each face f of S, and then connecting v_f by new edges to every vertex (primal or skew) on the boundary of f, a quadrangulation of $\mathbb{M}$ is obtained, in which every quadrangle has a primal, a skew, a dual, and a skew vertex, in this order, along its boundary.

3.3 Comparison of the two partitions

Information contained in a Morse-Smale complex may be divided in three parts: geometry (position of critical points and separatrix lines), topology (adjacency and incidence relations between cells of the complex), and elevation (function values at points of $\mathbb{M}$). Information contained in a subdivision has only two parts: geometry and topology. Here, we are interested in the topological part of the information content of the two partitions introduced above. We show that a Morse-Smale complex and a pair of overlayed dual subdivisions have basically the same topological (structural) properties.

Descending (ascending) Morse complexes of a Morse-Smale function f may be viewed as a subdivision S, in which minima (maxima) correspond to vertices, maxima (minima) correspond to faces, and saddles correspond to edges of S.

A Morse-Smale complex of f may be viewed as an overlay of S and its dual S^*, where primal, dual and skew vertices correspond to minima, maxima and saddles.

Conversely, a subdivision S may be viewed as a decomposition of $\mathbb{M}$ into descending (ascending) cells of maxima (minima) of some Morse-Smale function f. A pair of dual subdivisions may be viewed as an overlay of descending and ascending Morse complexes of f, where saddle points are at the intersection of primal and dual edges of S.

If f is a Morse (and not a Morse-Smale) function, then it may happen that a separatrix line connects two saddles, and there is no straightforward way to model an ascending (descending) Morse complex of f using a subdivision.

4 Refinement and Simplification Operations

We describe two analogous modifications of the structures we reviewed above, that is, on subdivisions and Morse-Smale complexes. Then we point out the connection between these operations on different structures, and we show that they are basicaly the same operation with different interpretation.

4.1 Operations on a Morse-Smale complex

The topology of a Morse-Smale complex can be simplified by an elementary operation which removes a pair of adjacent critical points, together with all the incident edges, and reconnects the remaining points, thus transforming one Morse-Smale complex into another, with fewer number of vertices. This operation is usually called the *cancellation of critical points*. The pair of removed critical points consists of a saddle and a maximum, or a sadle and a minimum. Thus, a cancellation does not change the Euler characteristic $\chi(\mathbb{M})$ of $\mathbb{M}$, which can be expressed using the number of maxima, saddles and minima of a Morse-Smale function f defined over $\mathbb{M}$ as $\chi(\mathbb{M}) = |maxima| - |saddles| + |minima|$. The removal of the saddle s and a minimum p, together with all the incident edges, from the initial Morse-Smale complex, illustrated in Figure 4 (top left), leaves the situation illustrated in Figure 4 (top right).

The points at the other end of the removed edges are affected by this removal, and they need to be reconnected. A saddle s_i, which was adjacent to p, will be connected by an edge to the minimum q. Intuitivelly, it can be imagined that both points s and p are moved towards q, and finally identified with it. Simultaneously, edges incident at s are deleted, and the edges (p, s_i), incident at p and connecting it to the remaining saddles s_i at the boundary of its ascending 2-cell, are extended to the edges (q, s_i), connecting those saddles to q. In other words, edge (q, s_i) is obtained by geometrically concatenating edges (q, s), (s, p) and (p, s_i), while keeping the edges (q, s_i) topologically separated, as illustrated in Figure 4 (bottom left). For better visual inspection, in Figure 4 (bottom right) the same Morse-Smale complex is represented, with edges

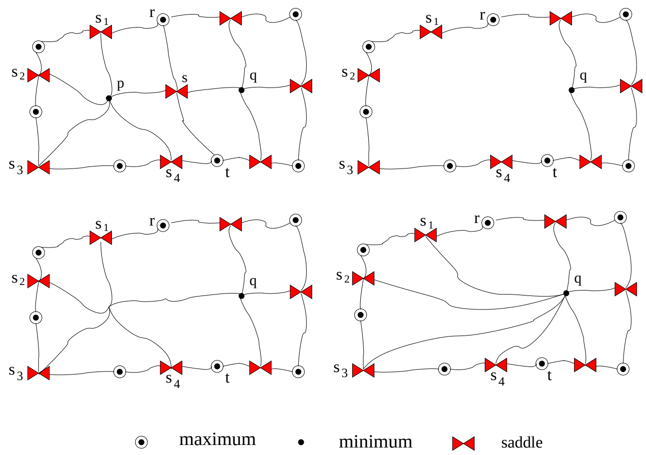

Figure 4: (top left) An initial Morse-Smale complex MS. (top right) MS with saddle s, minimum p, and all incident edges removed. (bottom right) Resulting Morse-Smale complex after the reconnection of saddles s_i to the other minimum q by concatenating paths. (bottom right) Resulting Morse-Smale complex with edges drawn apart.

(q, s_i) drawn apart. Note that the cancellation operation described above is not feasible (cannot be applied) to a saddle s and a minimum (maximum) p, if the region is of the type illustrated in Figure 2 (middle) (Figure 2 (right)).

The inverse of the cancellation operation is called anticancellation. It introduces a pair of critical points, consisting of a minimum (maximum) p and a saddle s (thus, $\chi(\mathbb{M})$ is invariant), and the corresponding edges, in such a way that the correct structure of the Morse-Smale complex is maintained. This means that, together with p and s, the two maxima (minima) t and r, which will become adjacent to s after anticancellation, have to be specified. It is illustrated in Figure 4 viewed in the reverse order.

A feasible (anti)cancellation does not change the structural properties of a Morse-Smale complex, or the Euler characteristc $\chi(\mathbb{M})$ of $\mathbb{M}$. When an (anti)cancellation is applied to a Morse-Smale complex, each saddle remains connected to two minima and two maxima, each region remains a quadrangle, and so on. The only restriction on the application of a feasible (anti)cancellation is elevation induced, i.e., when choosing a pair of critical points for an (anti) cancellation, care has to be taken that each newly introduced path between a saddle and a maximum (minimum) is ascending (descending).

4.2 Euler Operations on Subdivision

Euler operations derive the name from the fact that they maintain the Euler formula $V - E + F = \chi(\mathbb{M})$ which interrelates the number V of vertices, E

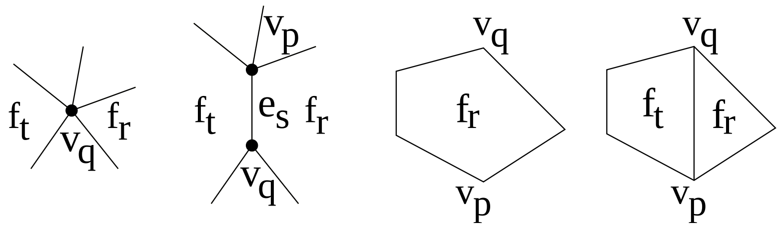

Figure 5: (Anti)cancellation of maxima interpreted as $(make)killFaceEdge$ (left), and (anti)cancellation of minima interpreted as $(make)killVertexEdge$ (right).

of edges and F of faces of a subdivision S of $\mathbb{M}$ with the Euler characteristic $\chi(\mathbb{M})$ of $\mathbb{M}$. Euler operations do not change the topological type of $\mathbb{M}$, but they change the number of elements (vertices, edges and faces) of a subdivision. There are four Euler operations, two of which involve vertices and edges, and the other two involve faces and edges.

For $makeFaceEdge$, a face f_t and two vertices v_p and v_q incident at f_t have to be specified. It makes a new face f_r and an edge e_s such that v_p and v_q are initial and terminal vertices of e_s, respectively, and f_t and f_r are faces left and right from e_s, respectively. This operation increases the number of faces and the number of edges of S by one. $killFaceEdge$ is the inverse of $makeFaceEdge$. A directed edge e_s (such that $e_sLeft \neq e_sRight$) has to be specified. When $e_sLeft = e_sRight$ (when e_s is a bridge), this operation is not feasible because it would disconnect S (and thus $\mathbb{M}$), and change the topological type of $\mathbb{M}$. This corresponds to an attempt to cancel a maximum in the region illustrated in Figure 2 (right) in a Morse-Smale complex. Edge e_s and its right face v_r are deleted, and its left face f_t substitutes f_r in all the edges incident at f_r. This operation decreases the number of faces and the number of edges of S by one. We illustrate $make(kill)FaceEdge$ in Figure 5 (left).

For $makeVertexEdge$, a vertex v_q and two faces f_t and f_r incident to v_q have to be specified. It makes a new vertex v_p and an edge e_s such that v_q and v_p are initial and terminal vertices of e_s, respectively, and f_t and f_r are faces left and right from e_s, respectively. This operation increases the number of vertices and the number of edges of S by one. $killVertexEdge$ is the inverse of $makeVertexEdge$. A directed edge e_s (such that $e_sOrg \neq e_sDest$) has to be specified. When $e_sOrg = e_sDest$ (when e_s is a loop), this operation is not feasible because it would merge the faces f_t and f_r into one, thus reducing to a $killFaceEdge$, i.e., to a cancellation of maxima and not of minima. This corresponds to an attempt to cancel a minimum in the region illustrated in Figure 2 (middle) in a Morse-Smale complex. Edge e_s and its terminal vertex v_p are deleted, and its initial vertex v_q substitutes v_p in all the edges incident at v_p. This operation decreases the number of vertices and the number of edges of S by one. We illustrate $make(kill)VertexEdge$ in Figure 5 (right).

4.3 Comparison of Modification Operations

If we interprete ascending and descending Morse complexes of a Morse-Smale function f as a subdivision S and its dual subdivision S^*, where minima, saddles and maxima correspond to vertices, edges and faces of S, respectively, then cancellation and anticancellation can be interpreted in terms of Euler operators on S.

Specifically, a cancellation of maximum r through a saddle s in t corresponds to merging two faces f_p and f_q into one by removing an edge from S, and thus to the operation $killFaceEdge(e_s)$, where e_s is the appropriately oriented edge (such that $e_sLeft = f_t$) of S associated with s. An anticancellation of r from t through s corresponds to spliting a face f_q in two faces, by adding an edge to S, specified by its endpoints v_p and v_q. It corresponds to the operation $makeFaceEdge(f_t, v_p, v_q)$ where f_t is the face associated with t, and v_p and v_q are the vertices of S associated with minima p and q which become adjacent to s after anticancellation. We illustrate (anti)cancellation of maxima (interpreted as $(kill)makeFaceEdge$) in Figure 5 (left).

Similarly, a cancellation of a minimum p and a saddle s in q may be viewed as merging of p and s in q or, equivalently, as edge collapse of the edge e_s associated with s. Thus, it corresponds to operation $killVertexEdge(e_s)$, where e_s is the appropriately oriented edge (such that $e_sDest = v_p$) of S associated with s. An anticancellation of p from q through s corresponds to vertex split of q, and the faces which become adjacent to the newly introduced edge e_s must be specified. Thus, an anticancellation corresponds to the operation $makeVertexEdge(v_p, f_r, f_t)$ where v_p is the vertex associated with p, and f_r and f_t are the faces of S associated with maxima r and t which become adjacent to s after anticancellation. We illustrate (anti)cancellation of minima (interpreted as $(kill)makeVertexEdge$) in Figure 5 (right).

Let us now highlight the benefit of interpreting operations on a Morse-Smale complex MS as operations on a subdivision S. In a region based approach, the result of an application of an operation on MS may be viewed as a replacement of an arbitrary set of 2-cells of MS with another set of 2-cells, which cover the same piece of $\mathbb{M}$, and have the same boundary, with the same set of edges and vertices. This approach takes into account the part of $\mathbb{M}$ covered by the removed cells, and does not take into account the highly regular and specific structure of a Morse-Smale complex. Interpretation of MS as a pair of dual subdivisions S and S^*, enables us to shift the attention from 2-cells to 1-cells (edges), that is, from region to boundary, and to use particular characteristics of the partition of $\mathbb{M}$ induced by MS. In a region-based approach, it is necessary to encode dependencies between modifications, based on the intersection of the regions covered by the 2-cells involved in the modifications, as in [1], where the region of influence of a cancellation of a saddle s is defined to be the region covered by four quadrangles which have s as one of the vertices. If, on the contrary, a Morse-Smale complex is regarded as a pair of dual subdivisions of $\mathbb{M}$, modifications can be expressed as operations on edges of S, and it becomes obvious that any two feasible modifications are independent, i.e., any two modifications can be performed in an arbitrary order, as long as all the newly introduced paths remain monotonic in elevation.

5 Concluding Remarks

A Morse-Smale complex is a data structure used for modeling of the morphology of a scalar function f. Starting from a full resolution Morse-Smale complex, and using simplification and refinement operations, an adaptive multiresolution representation of the complex can be obtained. Interpreting a Morse-Smale complex as a pair of dual subdivisions, a boundary based approach is addopted instead of a region based approach. It is seen that any two feasible modifications (refinement or simplification) of a Morse-Smale complex are independent, and can be performed in an arbitrary order. The only restriction which has to be imposed when performing the operations is elevation induced, that is, all the paths between a minimum and a saddle, and between a saddle and a maximum, have to be monotonic, while there are no restrictions related to the structure of a Morse-Smale complex.

In the future, we plan to investigate simplification and refinement operations on a 3-dimensional Morse-Smale complex.

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