Non-additive measures and integrals

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Abstract: There is presented a short overview on some results related the theory of non-additive measures and the corresponding integrals occurring in several important applications.

Key words and phrases: non-additive measure, aggregation function, Choquet integral, Sugeno integral, triangular conorm, triangular norm, pseudo-additive measure.

1 Introduction

Several types of integrals with respect to non-additive measures were developed for different purposes, [1, 5, 6, 15, 16]. We present some results related the theory of non-additive measures and the corresponding integrals important in several important applications. Many of these applications are related to functions defined on finite sets and therefore we restrict ourselves here on the finite case. We present also some results on special class of non-additive measures so called pseudo-additive (decomposable) measures and the corresponding integrals, which give a base for the so called pseudo-analysis. There are many important applications, for example in optimization problems, decision making, nonlinear partial differential equations, nonlinear difference equations, optimal control, fuzzy systems, [11, 12, 15, 16].

2 Non-additive measures

Let us consider $I = [0, 1]$ and $N = \{1, \dots, n\}$. A *set function* m on N is a function from 2^N to $\mathbb{R}$. A subset $A \subseteq N$ is equivalently denoted by $(\mathbf{1}_A, \mathbf{0}_{A^c}) \in [0, 1]^n$, or by its characteristic function $\mathbf{1}_A$ defined over N. We denote $\mathbf{x} = (x_1, \dots, x_n)$. Using the above equivalence, any set function m bijectively corresponds to a *pseudo-Boolean function* $f_m : \{0, 1\}^n \to \mathbb{R}$ by $f_m(x) = m(A_x)$ for all $\mathbf{x} \in \{0, 1\}^n$, where $A_{\mathbf{x}} = \{i \in N \mid x_i = 1\}$. Conversely, to any pseudo-Boolean function f corresponds a unique set function m_f such that

$m_f(A) := f(\mathbf{1}_A, \mathbf{0}_{A^c})$. Pseudo-Boolean functions are widely used in operations research. Cooperative game theory is devoted to a particular class of set functions, called *transferable utility games in characteristic form*. We will call them *games* or *non-additive measure* for simplicity. In the context of game theory, the set N is the set of players. A game $m : 2^N \to \mathbb{R}$ is a set function satisfying $m(\varnothing) = 0$. Useful examples of games are *unanimity games*. For any $A \subseteq N$, the *unanimity game* u_A on N is defined by:

$$u_A(B) := \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

Note that $u_\varnothing$ is not a game since $u_\varnothing(\varnothing) = 1$. A *capacity* $m : 2^N \to \mathbb{R}_+$ is a game such that $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$ (*monotonicity*). A capacity is *normalized* if $\mu(N) = 1$. Capacities are monotonic games, and were introduced originally by Choquet in 1953. They were rediscovered by Sugeno in 1974 under the name *fuzzy measure* [20].

Important connection with aggregation functions can be described in the following way, see [6]. Suppose we use as an aggregation function the weighted arithmetic mean

$$\mathsf{WAM}_{\mathbf{w}}(\mathbf{x}) = \frac{w_1 x_1 + \cdots + w_n x_n}{w - 1 + \cdots + w_n}$$

with respect to some weight vector $\mathbf{w} \in [0,1]^n$. It is easy to relate $\mathbf{w}$ to the values taken on by $\mathsf{WAM}_{\mathbf{w}}$, using particular vectors in $[0,1]^n$, namely $\mathbf{1}_i$: $\mathsf{WAM}_{\mathbf{w}}(\mathbf{1}_i) = w_i$ for all $i \in N$. This means that the value of function $\mathsf{WAM}_{\mathbf{w}}$ on $[0,1]^n$ is solely determined by its value at the endpoints of the n dimensions, which represents the weight of each dimension. In fact, the exact way $\mathsf{WAM}_{\mathbf{w}}(\mathbf{x})$, $\mathbf{x} \in [0,1]^n$, is determined from $\mathsf{WAM}_{\mathbf{w}}(\mathbf{1}_i)$, $i = 1, \ldots, n$, is linear interpolation. One may construct more complicated aggregation functions A by using more points in $[0,1]^n$ to determine A. A natural yet simple choice would be to take all vertices of $[0,1]^n$, namely $\{\mathbf{1}_A\}_{A \subseteq N}$. These include the previous endpoints of dimensions. Doing so, we have defined a set of weights $\{\mathbf{w}_A\}_{A \subseteq N}$, by

$$\mathsf{A}_{\mathbf{w}}(\mathbf{1}_A) = \mathbf{w}_A, \quad A \subseteq N.$$

It remains to construct $\mathsf{A}_{\mathbf{w}}$ on $[0,1]^n$ by some means (e.g., linear interpolation), using these points. By analogy with the previous case, $\mathbf{w}_A$ is the weight of the subset A of dimensions. In the case of $\mathsf{WAM}_{\mathbf{w}}$, the weight vector had no peculiar property, beside non-negativeness and normalization $\sum_i w_i = 1$. If weights are assigned to subsets of dimensions, then some properties are natural, especially if dimensions represent criteria or attributes, or individuals (voters, experts). In this framework, $\mathbf{x} \in [0,1]^n$ is a vector of scores, and $\mathsf{A}_{\mathbf{w}}(\mathbf{x})$ is the aggregated overall score, reflecting the score of each criterion or individual. Hence, $\mathsf{A}_{\mathbf{w}}(\mathbf{1}_A, \mathbf{0}_{A^c})$ is the overall score of an object having the maximal score for all criteria (individuals) in A and the minimal score otherwise, therefore the following properties are natural

(i) $\mathbf{w}_\varnothing = 0$, since the object $(\mathbf{1}_\varnothing, \mathbf{0}_N)$ is the worst possible;

(ii) $\mathbf{w}_N = 1$, since the object $(\mathbf{1}_N, \mathbf{0}_\varnothing)$ is the best possible;

(iii) $\mathbf{w}_A \leq \mathbf{w}_B$ whenever $A \subseteq B$, since object $(\mathbf{1}_B, \mathbf{0}_{B^c})$ is at least better on one dimension than $(\mathbf{1}_A, \mathbf{0}_{A^c})$.

Considering $\mathbf{w}$ as a set function on N, what we have defined above is nothing else than a *capacity*.

Let m be a set function on N, i.e., an element of $\mathbb{R}^{2^N}$. A *transform* is any mapping $T : \mathbb{R}^{2^N} \to \mathbb{R}^{2^N}$. The transform is *linear* if for any m_1, m_2 in $\mathbb{R}^{2^N}$ and any $\lambda_1, \lambda_2 \in \mathbb{R}$ it holds $T(\lambda_1 m_1 + \lambda_2 m_2) = \lambda_1 T(m_1) + \lambda_2 T(m_2)$, and it is *invertible* if T^{-1} exists. There are several useful invertible linear transforms of set functions. The best known one is the *Möbius transform*

$$\mu(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} m(B).$$

μ is said to be the *Möbius transform* (or *Möbius inverse*) of m. It is a linear and invertible transform, and $\mu(\varnothing) = m(\varnothing)$. The Möbius transform has been rediscovered many times. In the field of pseudo-Boolean functions, it appears as coefficients in the multilinear polynomial form of any pseudo-Boolean function f

$$f(\mathbf{x}) = \sum_{T \subseteq N} \left[a_T \prod_{i \in T} x_i \right] \quad (\mathbf{x} \in \{0,1\}^n).$$

In the field of cooperative game theory was found by Shapley in the form

$$\mu(A) = \sum_{B \subseteq N} m_B u_B(A) \quad (A \subseteq N),$$

i.e., any game (in fact, any set function) can be expressed in a unique way by unanimity games.

3 Integrals

We consider from now on the Choquet integral as an aggregation function over I^n. First, we consider non-negative vectors.

Definition 1 *Let m be a capacity on N, and $\mathbf{x} \in \mathbb{R}^n_+$. The Choquet integral of $\mathbf{x}$ with respect to m is defined by:*

$$\mathcal{C}_m(\mathbf{x}) := \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) m(A_{\sigma(i)})$$

with σ a permutation on N such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} := 0$, and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

It is straightforward to see that an equivalent formula is:

$$\mathcal{C}_m(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)}(m(A_{\sigma(i)}) - m(A_{\sigma(i+1)})),$$

with $A_{\sigma(n+1)} := \varnothing$. m need not to be monotone in order for the Choquet integral to be well defined, so that we could take as well a game instead of a capacity. Monotonicity of m is equivalent to monotonicity of the integral. Due to definition of aggregation functions as monotone function satisfying boundary conditions, in the case of aggregation functions we restrict to the case of capacities. The same remark applies to subsequent definitions as well. Similarly, only normalized capacities ensure that bounds of I are preserved, so that a Choquet integral is an aggregation function if and only if m is a normalized capacity.

The original definition, applicable to continuous spaces is the following.

Definition 2 *Let $f : \Omega \to \mathbb{R}_+$, and m a capacity on Ω. The Choquet integral of f with respect to m is defined by*

$$(C)\int f\,dm := \int_0^\infty m(\{\omega \in \Omega \mid f(\omega) > \alpha\})\,d\alpha.$$

The function $m(\{\omega \mid f(\omega) > \alpha\})$ is the *decumulative function* of m, non-increasing by monotonicity of m.

The case of real integrands leads to several definitions.

Definition 3 *Let $\mathbf{x} \in \mathbb{R}^n$, m be a capacity on N, and denote by $\mathbf{x}^+, \mathbf{x}^-$ the absolute values of positive and negative parts of $\mathbf{x}$, i.e., $\mathbf{x}^+ := \mathbf{x} \vee 0$ and $\mathbf{x}^- := (-\mathbf{x})^+$, where $\mathbf{0}$ stands for the null vector in $\mathbb{R}^n$.*

(i) The symmetric Choquet integral *of $\mathbf{x}$ with respect to m is defined by*

$$\check{\mathcal{C}}_m(x) := \mathcal{C}_m(\mathbf{x}^+) - \mathcal{C}_m(\mathbf{x}^-).$$

(ii) The asymmetric Choquet integral *of $\mathbf{x}$ with respect to m is defined by*

$$\mathcal{C}_m(x) := \mathcal{C}_m(\mathbf{x}^+) - \mathcal{C}_{\overline{m}}(\mathbf{x}^-).$$

The asymmetric integral is taken as the classical definition of the Choquet integral for real-valued functions, hence no special symbol is needed to denote it. It can be checked that if $\mathbf{x}$ is allowed to be in $\mathbb{R}^n$ in Definition 1, we get the asymmetric integral. This is not the case for the continuous formula of Definition 2. The symmetric integral has been proposed independently by Šipoš. Although apparently more natural, it leads to more complicated formulas. These two integrals take their name from the following property. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathcal{C}_m(-\mathbf{x}) = -\mathcal{C}_{\overline{m}}(\mathbf{x}) \quad \check{\mathcal{C}}_m(-\mathbf{x}) = -\check{\mathcal{C}}_m(\mathbf{x}).$$

3.1 The Sugeno integral

The Sugeno integral was introduced by Sugeno in 1972 [20], as a way to compute the expected value of a function with respect to a non-additive probability (called by Sugeno a "fuzzy measure", with the intention to give a subjective flavour to probability). Although mathematically very similar since, they differ only by their mathematical operators (sum and product being replaced by maximum and minimum respectively), it is more difficult to introduce in a natural way the Sugeno integral.

First, we consider non-negative vectors.

Definition 4 *Let m be a capacity on N, and $\mathbf{x} \in [0, \mu(N)]^n$. The Sugeno integral of $\mathbf{x}$ with respect to m is defined by*

$$\mathcal{S}_m(\mathbf{x}) := \bigvee_{i=1}^{n} \left[x_{\sigma(i)} \wedge m(A_{\sigma(i)})\right]$$

with σ a permutation on N such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} := 0$, and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$.

The above definition requires only comparison operators, no arithmetic ones. Hence, the definition works on any totally ordered set I, without additional structure.

We give the definition in the continuous general case [15, 20].

Definition 5 *Let m be a capacity on Ω, and $f : \Omega \to [0, m(\Omega)]$. The Sugeno integral of f with respect to m is defined by*

$$(S) \int f \, dm := \sup_{\alpha \in [0, m(\Omega)]} (\alpha \wedge m(\{\omega \mid f(\omega) > \alpha\}))$$

4 Pseudo-additive measures and the corresponding integrals

For the range of a set function instead of the field of real numbers in (Maslov, Samborski [12], Pap [13, 14, 15, 16], Sugeno, Murofushi [21]) it is taken a semiring on the real interval $[a, b] \subset [-\infty, +\infty]$, denoting the corresponding operations as $\oplus$ (pseudo-addition) and $\odot$ (pseudo-multiplication). This structure is applied for solving nonlinear equations (ODE, PDE, difference equations, etc.) using now the pseudo-linear principle (Litvinov, Maslov [11], Maslov, Samborski [12], Pap [14, 15, 16]). Based on semiring structure it is developed in [13, 14, 15, 16] the so called pseudo-analysis in an analogous way as classical analysis, introducing $\oplus$-measure, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. Here we shall restrict on the interval $[0, 1]$ and in this way pseudo-addition reduces on a triangular conorm and the pseudo-multiplication on a uninorm (or specially on a t-norm).

Definition 6 *Let* S *be a t-conorm. A mapping* $m : 2^N \to [0,1]$ *is called an* S*-measure (pseudo-additive measure, decomposable measure) if* $m(\varnothing) = 0, m(N) = 1$ *and for all* $A, B \in 2^N$ *with* $A \cap B = \varnothing$ *we have*

$$m(A \cup B) = \mathsf{S}(m(A), m(B)).$$

Based on the structure $([0,1], \mathsf{S}, \mathsf{U})$, where S is a continuous t-conorm and U a uninorm with the neutral element e and conditionally distributive with respect to S , there is developed in [8, 9] the so called (S, U)-integral.

Denote by $\mathcal{M}$ the set of all functions from N to $[0,1]$. As usual, a (step) function $f : N \to [0,1]$ is a function which assumes only finitely many values. If Range$(f) = \{a_1, a_2, \ldots, a_n\}$ with $a_i \neq a_j$ whenever $i \neq j$, and if $A_i = f^{-1}(\{a_i\})$, then there is a canonical representation of f given by

$$f = \mathop{\mathsf{S}}_{i=1}^{n} \mathsf{U}(a_i, \mathbf{1}_{A_i}^{\mathsf{S},\mathsf{U}}), \tag{1}$$

where

$$\mathbf{1}_A^{\mathsf{S},\mathsf{U}}(x) := \begin{cases} e & \text{for } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathsf{U}(a, \mathbf{1}_A^{\mathsf{S},\mathsf{U}}) = a\mathbf{1}_A$, where

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7 *Let* $m : 2^N \to [0,1]$ *be an* S*-measure.*

(i) Given a partition $\mathcal{C} = \{C_k \mid k \in K\}$ *the* (S, U)*-integral of a function* $f : N \to [0,1]$ *(which is represented as in (1)) is defined by*

$$\int_N^{(\mathsf{S},\mathsf{U})} f\, dm := \mathop{\mathsf{S}}_{k \in K} \Big(\mathop{\mathsf{S}}_{i=1}^{n} \mathsf{U}(a_i, m(A_i \cap C_k)) \Big).$$

(ii) The (S, U)*-integral of a function* $f : N \to [0,1]$ *over a set* $A \in 2^N$ *is defined by*

$$\int_A^{(\mathsf{S},\mathsf{U})} f\, dm := \int_N^{(\mathsf{S},\mathsf{U})} \mathsf{U}(\mathbf{1}_A^{\mathsf{S},\mathsf{U}}, f)\, dm.$$

The basic properties of (S, U)-integral are contained in [9].

For a function $f : N \to [a,b], -\infty \leq a < b \leq \infty$, the integral introduced in [12, 13] with respect to a semiring $([a,b], \oplus, \odot)$ can be reduced by a bijection $\varphi : [a,b] \to [0,1]$, choosing a suitable t-conorm S corresponding to the operation $\oplus$, on an (S, U)-integral with respect to an S-measure and a uninorm U or t-norm T corresponding to the operation $\odot$.

Example 1 (i) For $[a,b] = [0,\infty]$ we obtain the integral introduced in Sugeno, Murofushi [21], and for $([a,b], \oplus, \odot) = ([0,\infty], +, \cdot)$ we again come back to the classical integral.

(ii) If the operation $\oplus$ in the semiring $([a,b],\oplus,\odot)$ is not idempotent, then the operations $\oplus$ and $\odot$ are generated by some uniquely determined strictly increasing bijection $g:[a,b]\to[0,\infty]$ via

$$\begin{aligned} x\oplus y &= g^{-1}(g(x)+g(y)),\\ x\odot y &= g^{-1}(g(x)g(y)). \end{aligned}$$

The corresponding $(\oplus,\odot)$-integral was studied in Pap [13] (called g-integral in [14]), and it has the special form

$$\int_N^{\oplus} f\odot\, dm = g^{-1}\Big(\int_N (g\circ f)\, d(g\circ m)\Big),$$

where the integral on the right hand side is the classical integral. If $g\circ m$ is a probability on $N=\{1,2,\ldots,n\}$ and a function $f:N\to[0,1]$ is given by $f(i)=x_i, i=1,2,\ldots,n$, then the corresponding g-integral has the following form

$$\int_N^{\oplus} f\odot\, dm = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right),$$

which the weighted quasi-arithmetic mean with weights $w_i=g(m(\{i\}),$ such that $\sum_{i=1}^{n} w_i=1$.

5 The Benvenuti integral

Benvenuti integral is based on the chain representation (comonotone representation) of input vectors and two binary operations $\oplus$ and $\odot$, see [2]. For a constant $b\in\,]0,\infty]$, operation $\oplus:[0,b]^2\to[0,b]$ is supposed to be a continuous t-conorm, i.e., an associative continuous binary aggregation function with neutral element 0. For another constant $c\in\,]0,\infty]$ (the case $c=b$ is possible and most frequent case) operation $\odot:[0,b]\times[0,c]\to[0,b]$ is a non-decreasing binary operation which is right-distributive with respect to $\oplus$, i.e.,

$$(u\oplus v)\odot w=(u\odot v)\oplus(v\odot w)$$

for all $u,v\in[0,b]$ and $w\in[0,c]$. Moreover, define a binary operation $\ominus:[0,b]^2\to[0,b]$ associated to $\oplus$ by

$$u\ominus v=\inf\{t\in[0,b]\mid v\oplus t\geq u\}$$

(compare [22]).

Definition 8 *For a fixed $n\in\mathbb{N}$, let $m:2^N\to[0,c]$ be a monotone set function (capacity).* Benvenuti integral $B_m^{\oplus,\odot}:[0,b]^n\to[0,b]$ *is given by*

$$B_m^{\oplus,\odot}(\mathbf{x}):=\oplus_{i=1}^{n}(x_{(i)}\ominus x_{(i-1)})\odot m(A_{(i)}).$$

Many important special cases are in the following example.

Example 2 (i) Let $b = c = \infty, \oplus = +, \odot = \cdot$ on $[0, \infty]$. Then $B_m^{+,\cdot} = C_m$.

(ii) Let $b = c = 1, \oplus = \vee, \odot = \wedge$. Then $B_m^{\vee,\wedge} = S_m$.

(iii) Let $b = c = 1, \oplus = \mathsf{S}_\mathbf{P}$ (probabilistic sum), i.e., $u \oplus v = u + v - uv$, and $\odot : [0,1]^2 \to [0,1]$ is a uninorm generated by a multiplicative generator $\varphi : [0,1] \to [0,1]$ given by $\varphi(x) = -\log(1-x)$, i.e.,

$$u \odot v = \exp(-\log(1-u)\log(1-v)),$$

and the neutral element of $\odot$ is $e = 1 - \exp(-1)$. Then

$$B_m^{\oplus,\odot}(\mathbf{x}) = \varphi^{-1}\left(C_{\varphi o m}(\varphi \circ \mathbf{x})\right),$$

i.e., Benvenuti integral is a φ-transform of the Choquet integral.

(iv) Let $b = c = 1, \oplus = \vee, \odot = \cdot$. Then $B_m^{\vee,\cdot}(\mathbf{x}) = \max_i(w_i \cdot x_i)$, where $x_i = f(i)$, gives the Shilkret integral [17] (where the integral was considered with respect to $\mathsf{S}_\mathbf{M}$-measures).

For more details see [2].

6 Measure based aggregation functions

A general unified approach to fuzzy integrals is given in [10]. For any fuzzy measure μ defined on Borel subsets of the open unit square with uniform marginals, i.e.,

$$\mu(]0,x[\times]0,1[) = \mu(]0,1[\times]0,x[) = x$$

for all $x \in [0,1]$, the following functional was introduced: for m a fuzzy measure on 2^N and $f : N \to [0,1]$,

$$I_{\mu,m}(f) := \mu\big(\{(x,y) \in]0,1[^2 \mid y < m(f \geq x)\}\big) \tag{2}$$

If μ is a probability measure on Borel subsets of $]0,1[^2$, then it is in a one-to-one correspondence with a 2-copula C. Thus we can use also notation $I_{C,m}$ for the integral introduced in (2), and then it holds

$$I_{C,m}(f) = \sum_{i=1}^{n} \big(C(x'_i, m(f \geq x'_i)) - C(x'_i, m(f \geq x'_{i+1}))\big) \tag{3}$$

and equivalently also

$$I_{C,m}(f) = \sum_{i=1}^{n} \big(C(x'_i, m(f \geq x'_i)) - C(x'_{i-1}, m(f \geq x'_i))\big), \tag{4}$$

where x'_i is the i-th order statistics form the sample $(f(1), \ldots, f(n))$ and $x'_0 = 0, x'_{n+1} = \infty$ by convention.

For the product copula Π the corresponding integral I_{Π} is just the Choquet integral and formulas (3) and (4) are two equivalent forms for this integral on N given in [10]. Moreover, $I_{\mathsf{T_M}}$ is the Sugeno integral.

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