Digital convex fuzzy hull

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Abstract: Problem of defining convexity of a digital region is considered. Definition of $DL-$(digital line) convexity is proposed, and it is shown to be stronger than the other two definitions, $T-$(triangle) convexity and $L-$(line) convexity. In attempt to connect the convexity of digital sets, to the fact that digital set can be a fuzzy set, the notion of convexity of the membership function is introduced. Digital set is placed in (x, y)-plane, and values of the membership function for every centroid of that set is drifted on z-axis, so our consideration is fully and clearly represented in $\mathbb{R}^3$. For convexity in (x, y)-plane we chose the $DL-$convexity (digital line convexity) as the most appropriate for our purpose. For membership function the quasi convexity is chosen. All these notions are introduced, in order to be able to define the digital convex fuzzy hull of a digital fuzzy set.

Key words and phrases: convexity, $DL-$convexity, $DL-$convex hull, digital fuzzy set, digital convex fuzzy hull, fuzzy set.

1 Introduction

A set S in Euclidean space is convex if for any two points P and Q states:

$$\text{for any } \lambda \in [0,1],\ \lambda P + (1-\lambda)Q \in S.$$

The convex hull S_H of a set S is the intersection of all convex sets that contain S:

$$S_H = \bigcap_{i\in I}\{S_i : S \subset S_i, S_i \text{ is convex}\}.$$

After those definitions we can say that S is convex iff $S_H = S$.

Euclidean space is digitized using specific regular tessellation; each square is called a pixel. Set R (or region) in the digital space is the union of a set of pixels, and we call it a digital set. The centroid of a pixel is the point of intersection of the two diagonals of the pixel. The union of the centroids of the pixels of R noted by R', may be considered as the lattice point or simply a point representation of R. Conversely, R may be called the region representation of R'.

Considering convex hull of the given digital set, the most important issue is how to find algorithm for computing of appropriate convex hull. Finding Euclidean convex hull for the given region is rather trivial, so it might be used. Further, we could redigitize that set. Some authors treated this problem by discussing different kinds of convexity, but they did not get appropriate convex hull. So, in the next section, we will present a short overview of this kinds of convexity and their deficiencies and try to overcome them by defining a new kind of convexity which gives us more attractive results in computing the convex hull of a region.

2 $DL-$convexity

Definition 1 *A pixel q is a $4-$neighbor of a given pixel p, if p and q share an edge.*

Definition 2 *A pixel q is an $8-$neighbor of a given pixel p, if q and p share either an edge or a vertex.*

Definition 3 *A set of pixels P is $4-$connected if for every pair of pixels p_i and p_j in P there exist a sequence of pixels p_i,...,p_j such that:*
a) all pixels in the sequence are in the set P;
b) every 2 pixels that are adjacent in the sequence are $4-$neighbors.

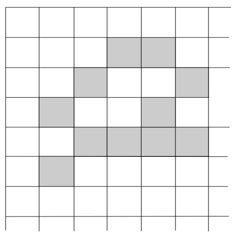

Figure 1

Definition 4 *A set of pixels P is $8-$connected if for every pair of pixels there exists a sequence of pixels p_i, ..., p_j such that:*
a) all pixels in the sequence are in the set P;
b) every 2 pixels that are adjacent in the sequence are $8-$connected.

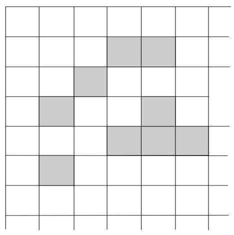

Figure 2

In Figure 1 is $8-$connected area, and in Figure 2 is disconnected area.

Definition 5 *Let R be a simply $8-$connected digital region with its point representation given by R'. We define on R the closed region R'_t which consists of all points z (not necessary lattice points) that lie in triangles $\triangle p_1p_2p_3$ where p_1, p_2 and p_3 are in R' and mutually $8-$neighbors if distinct.*

The boundary of $R_t^{'}$ is denoted by $\delta R'$.

Definition 6 *For any two points p_1,p_2 of R', $P(R', p_1, p_2)$ denotes the set of polygons whose boundaries consist of nonempty subsegments of the line segment p_1p_2 and $\delta R'$ and whose interiors are subsets of R_t'.*

Kim [5] defined few definitions which can be useful:

Definition 7

- *Line property: A digital region R' possesses the line property iff there are no three colinear digital points (p_1, p_2, p_3) such that p_1 and p_3 belong to R' but p_2 belongs to $\overline{R'}$;*
- *Chord property: R' has the chord property iff for any (Euclidean) point (x, y) on p_1p_2, where p_1, p_2 belong to R', there is a digital point $p = (h, k)$ of R' such that*

$$\max\{|x - h|, |y - k|\} < 1;$$

- *Area property: A digital region R' has the area property iff for any two points p_1 and p_2 of R', $P(R', p_1, p_2)$ has no polygon that contains points of $\overline{R'}$.*

Kim defined a digital region to be convex if it is the digitization of a (Euclidean) convex region. He showed that these three properties are equivalent to convexity if R' is simply $8-$connected.

Definition 8

- *Hull property: If $H(R')$ is the Euclidean convex hull of the points of R' then R' is said to have hull property if $H(R')$ does not contain any point of $\overline{R'}$;*
- *Digital line property: R' is said to have digital line property if for any two points p_1 and p_2 in R' some digital straight line segment between p_1 and p_2 belongs to R';*
- *Triangle property: R' is said to have triangle property if for any three points p_1, p_2, p_3 of R', the digital points in triangle $p_1p_2p_3$ all belong to R'.*

The first property was derived by Kim, the second by Kim and Rosenfeld and the third by Ronse and every one of them is equivalent to the corresponding property from the previous definition. Ronse showed that triangle property is equivalent with hull property and defined $L-$convexity and $T-$convexity based on the line property and area property, respectively. Ronse defined the $T-$convex hull of a region R' as the set of cellular points belonging to $H(R')$ and tried to solve problem of computing the convex hull using this definition.

However, it can be shown that $T-$convex, or $L-$convex region may not be $8-$connected, or may be $8-$connected, but not $4-$connected, and that

$T-$hull of a region may also be disconnected in certain situations when R' has several connected components. The possibility of a digitally convex region being disconnected is not intuitively appealing since it does not fit with the concept of convexity in Euclidean space. Hence, we propose another definition of digital convexity for which the digital convex hull is always $8-$connected.

Definition 9 *Digital straight line segment AB is the set of all pixels that are intersected by Euclidean straight line segment AB, where A, B are digital points (centroids).*

Definition 10 *A digital region R' is called $DL-$convex if for any two digital points p_1, p_2 belonging to R', there is a digital straight line segment between p_1 and p_2 whose points all belong to R'.*

A digital straight line may be $4-$connected or $8-$connected. Thus, we can have either $4DL-$convexity or $8DL-$convexity. However, $8DL-$convexity suffices for our purpose.

Theorem 1 *A $DL-$convex region is simply connected, i.e. it is connected and does not contain any holes.*

Theorem 2 *If R' is $DL-$convex it is $T-$convex as well as $L-$convex, but the converse is not generally true.*

The next theorem is the most important for our purpose because it is used for the algorithm for computing DL-convex hull (see Chaudhuri, Rosenfeld [3]):

Theorem 3 *If R' is $T-$convex (or $L-$convex) and $8-$connected it is $DL-$convex.*

Definition 11 *$H_{DL}(R')$, the $DL-$hull of R' is the minimal $DL-$convex set that contains R'.*

In other words, no proper $DL-$convex subset of $H_{DL}(R')$ can contain R'. The main problem, now, becomes how to compute $DL-$convex hull of a given digital region, for which the equivalence of $T-$convexity and $DL-$convexity of an $8-$connected figure gives an attractive solution in most cases.

3 The convexity of fuzzy sets

Dealing with digital sets, we found it interesting to define fuzzy sets whose universal set will be digital sets, and then to consider the convexity of these sets. In accordance to this, let us represent some basic definitions and theorems related to fuzzy sets and their convexity, and extend them on digital sets.

Definition 12 *A fuzzy set A defined on the universal set X is a set of ordered pairs: $A = \{(x, \mu_A(x))|x \in X\}$ where $\mu_A : X \to [0, 1]$ is the membership function of A in X.*

Definition 13 *The support of a fuzzy set A is the set*

$$\text{supp}(A) = \{x \in X|\mu_A(x) > 0\}.$$

Definition 14 *An α-cut of a fuzzy set A, for $\alpha \in [0, 1]$, is the set:*

$$A_\alpha = \{x \in X|\mu_A(x) \geq \alpha \ \}.$$

Definition 15 *A fuzzy set A of a set $\mathbb{R}^n$, given by the membership function μ_A, is called quasiconvex (see [1], [11]) if*

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

holds for every $x_1, x_2 \in \text{supp}(A)$, and $\lambda \in [0, 1]$.

Theorem 4 *The fuzzy set is quasiconvex if and only if its α-cuts are empty or convex sets for each $\alpha \in [0, 1]$.*

Definition 16 *A fuzzy subset A of a set $\mathbb{R}^n$, given by the membership function μ_A, is called convex if*

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda\mu_A(x_1) + (1 - \lambda)\mu_A(x_2)$$

holds for every $x_1, x_2 \in \text{supp}(A)$, and $\lambda \in [0, 1]$.

This means convexity in classical sense and discerns with quasiconvexity. For example, fuzzy subset A of $\mathbb{R}$, given by its membership function $\mu_A(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$ is a quasiconvex fuzzy set, but it is not convex. It also holds that α-cut of a convex fuzzy set is convex (or empty set) for every $\alpha \in [0, 1]$, but the converse does not hold in general.

Definition 17 *A convex fuzzy hull, $fconv(A)$, of a fuzzy set A is defined as the smallest convex fuzzy set whose support is A.*

It is rather trivial that convex fuzzy set is equal to the corresponding convex fuzzy hull, and that convex fuzzy hull of a given set A is, in fact, the standard fuzzy intersection of all convex fuzzy sets that contain A.

In [12] a fuzzy convex combination is defined.

Definition 18 *For a given fuzzy subset A of $\mathbb{R}^n$, convex fuzzy combination of elements $x_1, x_2, ..., x_n \in \mathbb{R}^n$ is the smallest fuzzy set C_A such that:*

$$\mu_{C_A}(x) \geq p_1\mu_A(x_1) + p_2\mu_A(x_2) + ... + p_n\mu_A(x_n),$$

for all $x = p_1x_1 + p_2x_2 + ... + p_nx_n$, where real numbers $p_i \geq 0$, for $i = 1, 2, ..., n$ are such that $\sum\limits_{i=1}^{n} p_i = 1$.

Theorem 5 1) *It is shown that a fuzzy set A is convex iff for every integer $n \geq 1$, every convex fuzzy combination of any n points of A is included in A.*

2) *The convex fuzzy hull of a fuzzy set A is equal to the union of all convex fuzzy combinations of A.*

Theorem 6 *The intersection of any family of convex fuzzy sets is a convex fuzzy set, while the union of a family of convex fuzzy sets is not necessarily a convex fuzzy set (see [2]).*

Definition 19 *A fuzzy set A of $\mathbb{R}^n$ is called fuzzy $\alpha-$ convex for some $\alpha \in [0,1]$ if*

$$\mu_A(ax_1 + (1-a)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)),$$

holds for all $x, y \in A$.

Definition 20 *A digital fuzzy set is a fuzzy set which is defined on some digital universal set, i.e. on its point representation.*

Definition 21 *We define digital convex fuzzy hull of a digital fuzzy set $\mathcal{A}$ of $\mathbb{Z}^2$, given by the membership function μ, as the smallest fuzzy set $\mathcal{H}_{DLF}(\mathcal{A})$, such that*

$$\operatorname{supp}(\mathcal{H}_{DLF}(\mathcal{A})) = H_{DL}(\operatorname{supp}(\mathcal{A})),$$

$$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

holds for every $x_1, x_2 \in H_{DL}(\operatorname{supp}(\mathcal{A}))$, and for all $\lambda \in [0,1]$ such that $\lambda x_1 + (1-\lambda)x_2 \in \mathbb{Z}^2$.

All these notions and the properties related to them are given in order to be able to make an algorithm for computing digital convex fuzzy hull. This algorithm is presented in the paper [9].

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References

[1] N. E. Ammar, Some Properties of Convex Fuzzy Sets and Convex Fuzzy Cones, Fuzzy Sets and Systems 106, 381-386, 1999.

[2] E. Ammar, J. Metz, On Fuzzy Convexity and Parametric Fuzzy Optimization, Fuzzy Sets and Systems 49, 135-141, 1992.

[3] B. B. Chaudhuri, A. Rosenfeld, On the Computation of the Digital Convex Hull and Circular Hull of a Digital Region, Pattern Recognition, Vol. 31, No. 12, pp. 2007-2016, 1998.

[4] L. Čomić, N. M. Ralević, On Convex Hulls of Fuzzy Sets, 6^{th} International Symposium Interdisciplinary Regional Research, Hungary-Romania-Yugoslavia, Novi Sad 3-4. oktobar 2002.

[5] C. E. Kim, On the cellular convexity of complexes, IEE Trans. Pattern Analysis Mach. Intell. PAMI-3,617-625, 1981.

[6] C. E. Kim, A. Rosenfeld, Digital straight lines and convexity of digital regions, IEE Trans. Pattern Analysis Mach. Intell. PAMI-4, 149 – 152, 1982.

[7] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[8] N. M. Ralević, Digitals objects and convex fuzzy sets, Acta Mechanica Slovaca (accepted).

[9] N. M. Ralević, V. Ćurić, M. Janev, Algorithm for computing the digital convex fuzzy hull (accepted).

[10] C. Ronse, Definition of convexity and convex hulls in digital images, Bull. Soc. Math de Belgique 37, 71-85, 1985.

[11] X. Yang, A Note on Convex Fuzzy Sets, Fuzzy Sets and Systems 53, 117-118, 1993.

[12] X. Yang, Some Properties of Convex Fuzzy Sets, Fuzzy Sets and Systems 72, 129-132, 1995.