Practical Applications of the Hodographic Approximation Method on Hydrodynamic Specialized Networks

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Abstract: Based on the results of [3] and [4] the current paper presents a general method which can be applied for any approximation type of the hodographic method to study the compressible fluid's permanent and subsonic flux through profiles' networks. Based on the defined mathematical model an analytical method is searched by using the linear approximation of the compressible fluid's characteristic curve and the special network's turbine profile. To be able to analyze on all the three hodographic methods the flux of the compressible fluid through the special network the method of C. C. Lin (1949) was generalized. The method can be used even if the elements of the special's network are not turbine profiles, but the obstacles are satisfying the requirements of a geometrically and phisically periodic system.

Keywords: hodographic variables, hodographic approximation method, hydrodynamic network, complex potential motion, function of J. Leray

1 Introduction

Lot of practical engineering problems can be originated from the solution of boundary value problems. In most of the implied problems in engineering analysis, the real domain of the boundary value problems has irregular boundaries, with complex properties of the domain from one zone to the other, which are excluding any possibility to find analytical solutions for the fundamental equations. In this case, the modern numerical methods represent the only way to obtain the suitable solutions. They were used with a division of the complex domain by a grid, such as the method of finite elements (MFE), or by the division of the domain in finite elements (finite difference method, MDF), or by linearly approximating the physical model and its coupled geometrical model. The Boundary Element Method (BEM) is an alternative new method of numerical study, where only the boundary of the analyzed domain is divided in finite elements thus obtaining fewer elements that in MFE. As a result, BEM proved to be very effective in economical and engineering boundary problems. The main point of the real-BEM [1], is the determination of the fundamental integral equation of the solution in a domain, with the aid of the values of the solutions on the boundary and of the "flux values". By aid of this formula the integral equation on the boundary domain is written, and by discretization of the integral equation on boundary, the algebraically equation system which result in the discretized solution gives a boundary. The engineering applications of this method were discussed in [2].

In the present paper a practical linear approximation is presented which is capable to solve the compressible fluid's flux through profiles' network by using all three possibilities of the known hodographic methods (Tschiaplighin-Demtchenko version, Kármán-Tschien version, Caius Iacob version). The presented method searches an analytical solution and for this linear approximation of the physical model (the compressible fluid's characteristic curve) and of the geometrical model (special network's turbine profile) is applied.

2 Presenting the Analyzed Physical and Geometrical Models

The application of the profile grid theory has an important place in the design and improvement of the modern turbo machines turbines'. The mathematical models used in the profile grid modern theory consider the structure and physical characteristics of real fluid-course. To understand the hydrodynamics of the special network let us consider an axial flow surface around the turbines. In the domain of the turbine's blade cylinder-type flux surfaces are created. If such a surface is cut by a blade and this is projected on a plane, a plane specialized network is obtained, containing a finite number of blade (obstacles) profiles. To generalize the profile grid theory, the finite numbers of blade profiles are substituted with infinite ones, where the elements of the blade profiles (obstacles) are repeated periodically. Bigger number of the turbine blades approximates better the real flux.

Definition 1: *A periodic coplanar system of specialized plane obstacles is defined as a plane specialized network.*

Definition 2: *The straight line crossing the collinear point of the specialized plane obstacles, which creates a λ angle with the l length chord is defined as the director line (ax) of the specialized network (Fig. 1).*

Definition 3: *The period of the network* $\overline{\omega} = t \cdot e^{i(\frac{\pi}{2} - \lambda)}$ *is the quantity that should be used for moving a profile on the director line in order to achieve a neighboring profile (t – the scale of the network).*

The plane specialized network is adjusted to the $z = x + iy$ coordinating system of a complex space, where the $0x$ ax is parallel with the specialized plane obstacles. The curves delimiting the specialized plane obstacles are notated L_k ($k \in Z$), while the inner area of the L_k specialized plane obstacle in the t-width periodicity zone is D_k^+, the extern domain is D_k^-. In the followings, a compressible fluid flux is analyzed, with $\vec{V}_1$ velocity flux at $-\infty$ and $\vec{V}_2$ velocity flux $+\infty$.

Definition 4: *The principal periodical strip* D_0^- *is the extern domain of the profile with* L_0 *base, situated in the strip whose width is t.*

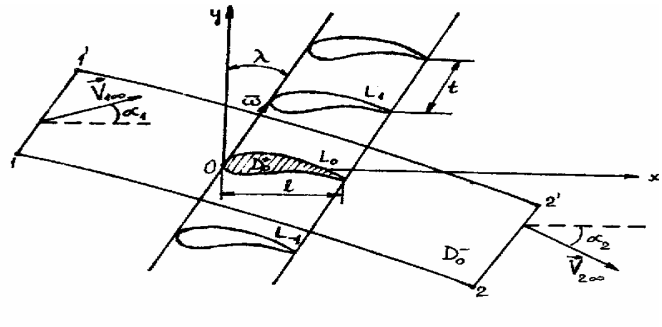

Figure 1
The director line of the specialized network

Remark 1: Due to the fact that the hydrodynamic specialized network is physically and geometrically periodic, it is enough to know in the principal periodical D_0^- the motion of the compressible fluid.

Property 1: The motion around the specialized network in the z complex plane is the result from a source (Q, Γ_1) placed at $-\infty$, pointing to another source (Q, Γ_2) which is placed at $+\infty$.

Of course, $\Gamma = \Gamma_1 - \Gamma_2 - \ldots -$ is the magnitude of the circulation around L_0 profile.

Remark 2: Using the kinematical and geometrical parameters, one can calculate the motion's hydrodynamic parameters from the following equations [1], [11]:

$$\Gamma_1 = V_1 t \sin(\lambda + \alpha_1) \qquad \Gamma_2 = V_2 t \sin(\lambda + \alpha_2) \qquad \Gamma = \Gamma_1 - \Gamma_2$$

$$Q = \varsigma_1 \cdot t \cdot V_1 \cos(\lambda + \alpha_1) = \varsigma_2 \cdot t \cdot V_2 \cos(\lambda + \alpha_2) \tag{1}$$

where V_1 and V_2 represents the asymptotic velocity, while α_1 and α_2 the angle closed with these velocities and the *0x* ax.

To analyze such a complex physical and geometrical model, two methods are known:

a) Integral equation method or hydrodynamic singularities method. This method was used to develop a calculus algorithm [2], [10], using the p-analytical complex function theory and the boundary elements numerical method.

b) Hodographic method [9]: difficult and complex calculation is applied, but the solution gives an analytical solution.

3 Practical Use of the Approximating Hodograph Method

In the $z(x,y)$ complex plane of the compressible fluid consider a specialized network with infinite number of specialized plane obstacles (scaling parameter t, considered angle λ).

We consider known the fluid's velocity at $-\infty$: $\vec{V}_1 = V_1 \cdot e^{i\alpha_1}$, while at $+\infty$ the velocity becomes $\vec{V}_2 = V_2 \cdot e^{i\alpha_2}$.

The motion of the fluid inside the network is given by the following equations:

- Continuity (Euler equation):

$$\operatorname{div}\left(\rho \cdot \vec{V}\right) = 0 \quad \text{or} \quad \frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) = 0 \tag{2}$$

- The equation of the state of the motion:

$$p = p(\rho) \quad \text{or} \quad \frac{p}{\rho^{\gamma}} = const, \quad \gamma = \frac{c_p}{c_v} \tag{3}$$

- The equation of the irotational motion potential:

$$\operatorname{rot} \vec{V} = 0 \quad \text{or} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4}$$

where $u = V_x$, $v = V_y$ are the components of the $\vec{V}$ velocity, c_p and c_v are special temperature values at isobar (constant pressure) and izochor (constant temperature) conditions.

From (2) and (4) it can be deduced that in the occupied D region $(\exists)\ \varphi,\ \psi \in C^2(D)$ which functions are satisfying the following system of equation:

$$\frac{\partial\varphi}{\partial x} = \frac{\rho_0}{\rho}\cdot\frac{\partial\psi}{\partial y},\quad \frac{\partial\varphi}{\partial y} = -\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x},\quad u = \frac{\partial\varphi}{\partial x},\quad v = \frac{\partial\varphi}{\partial y} \tag{5}$$

where ρ_0 represents the null-velocity of D.

Due to the fact, that (5) is not the solution of the linear partial derivative equation system, in the literature a lot of other linearizing solutions were published [1], [9]. S. A. Tschiaplighin was the first who proposed to switch the quasi-linear system (5) to a linear equation system, based on the independent hodographic variables (V,θ):

$$V\cdot e^{-i\theta} = u - iv \qquad \text{or} \qquad u = V\cos\theta,\ \ v = V\sin\theta \tag{6}$$

Property 2 [1]**:** The relation between the physical and hodographic plane can be written by the following complex variable equation:

$$dz = \frac{e^{i\theta}}{V}\left(d\varphi + i\frac{\rho_0}{\rho}d\psi\right),\ z = x + iy \tag{7}$$

Property 3 [1]**:** The description of the compressible fluid's motion in the (V,θ) hodographic plane is given by the following equation:

$$\begin{cases} \dfrac{\partial\theta}{\partial\varphi} = \dfrac{\rho}{\rho_0 V}\dfrac{\partial V}{\partial\psi} \\ \dfrac{\partial\theta}{\partial\psi} = V\left(\dfrac{\rho_0}{\rho V}\right)\dfrac{\partial V}{\partial\varphi} \end{cases} \tag{8}$$

Property 4 [9]**:** With the B. Demtchenko function substitution:

$$\sigma = \sigma_1 + \int_{v_1}^{v}\frac{\rho}{\rho_0 V}dV \tag{9}$$

the equation system (8) becomes:

$$\frac{\partial\theta}{\partial\varphi} = \frac{\partial\sigma}{\partial\psi},\qquad \frac{\partial\theta}{\partial\psi} = -K(V)\frac{\partial\sigma}{\partial\varphi} \tag{10}$$

where:

$$K(V) = \frac{\rho_0 V^2}{\rho}\left(\frac{\rho_0}{\rho V}\right)'_V \tag{11}$$

The hypothesis of S. A. Tschiaplighin was in case of the subsonic motion, by other words the velocity of the fluid is smaller than half of the sound velocity ($V < c/2$) then $K(V) \cong 1$. Starting from this observation, the approximation of $K(V)$ is:

$$K(V) = -\frac{\rho_0 V^2}{\rho}\left(\frac{\rho_0}{\rho V}\right)'_V = 1 \tag{12}$$

As a result, there exists a fictitious fluid (based on the Tschiaplighin-hypothesis) with the following compressible law:

$$\frac{\rho_0 V^2}{\rho}\left(\frac{\rho_0}{\rho V}\right)'_V = -1 \tag{13}$$

This fluid gives the possibility to substitute the nonlinear characteristic equation (3) with the (3^*) linear characteristic equation:

$$p = C\frac{1}{\rho} + C' \tag{3*}$$

where the C and C' constants can be determined based on the used approximation and it reflects the real physical conditions. In this way, a hodographic approximation method in the $\left(\frac{1}{\rho}, p\right)$ plane substitutes the (3) isentropic curve with (3^*).

Property 5 [9]**:** Choosing the C and C' constants of the (3^*) equation three possibilities exist. These are named as the hodographic equation variants (Fig. 2).

The demonstration of Property 5 was done by S. Popp in 1969 [9]. It resulted that only three hodographic approximation variant methods exists (Fig. 2):

- The Tschiaplighin-Demtchenko approximation (line nr. 1);
- Kármán-Tschien approximation (line nr. 2);
- Caius Iacob approximation (line nr. 3).

C. C. Lin was the first [6], who has demonstrated in 1949 that in case of a compressible fluid's flux around the specialized plane obstacles, the Kármán-Tschien approximation can be utilized.

Property 6 [9]**:** Using the Tschiaplighin-type fictitious fluid, the following linear partial equation system can be obtained:

$$\frac{\partial\theta}{\partial\varphi} = \frac{\partial\sigma}{\partial\psi}, \quad \frac{\partial\theta}{\partial\psi} = -\frac{\partial\sigma}{\partial\varphi} \tag{8*}$$

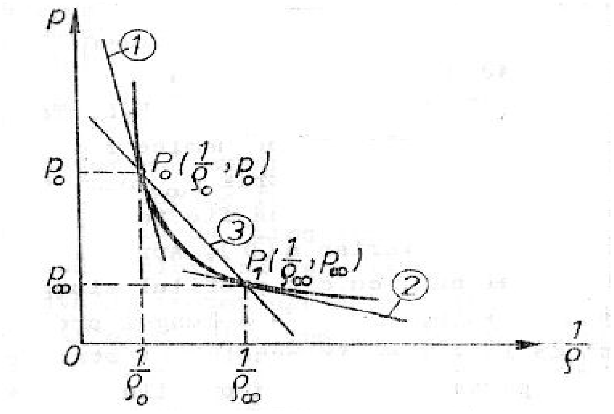

Figure 2
The director line of the specialized network

Remark 3: Equation (8*) reflects that the $\omega = \theta + i\sigma$ complex function is an analytic function of the $f = \varphi + i\psi$ complex variable. The $\omega = \theta + i\sigma$ function is the Levi-Civita function of the (ζ) fictitious plane's incompressible fluid, who's motion complex potential is: $f = \varphi + i\psi$.

Theorem 1 [1]**:** In the hodographic approximation method, the correspondence between the uncompressible circulational motion from plane (ζ) and the subsonic compressible motion from plane (*z*) is given by the following equations:

$$z = C_1 \int h(\zeta) d\zeta + C_2 \int \overline{\left(\frac{df}{d\zeta}\right)^2 \frac{d\zeta}{h(\zeta)}} \tag{14}$$

$$\frac{1}{V} = \frac{C_1}{|w|} + C_2 |w| = \frac{C_1}{V_i} |h(\zeta)| + C_2 \frac{V_i}{|h(\zeta)|} \tag{15}$$

$$\frac{\widetilde{\rho}_0}{\rho V} = \frac{C_1}{|w|} - C_2 |w| = \frac{C_1}{V_i} |h(\zeta)| - C_2 \frac{V_i}{|h(\zeta)|} \tag{16}$$

$$\theta = \theta_1 + \arg h(\zeta) \tag{17}$$

where:

- V_i is magnitude of the uncompressible fluid's velocity;
- $\widetilde{\rho}_0$ is the fictitious the uncompressible fluid's density;
- C_1 and C_2 constants – in case of each approximation variants – are determined by the following expressions:

- Tschiaplighin – Demtchenko approximation variant:

$$C_1 = \frac{1}{2}\left(1+\sqrt{1+M_0^2}\right), \quad C_2 = -\frac{1}{2V_\infty^2}\left(1-\sqrt{1+M_0^2}\right) \tag{18}$$

where $M_0 = \frac{V_\infty}{C_0}$ is the Tschiaplighin-number.

- Kármán-Tschien approximation variant:

$$C_1 = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-M_\infty^2}}\right), \quad C_2 = -\frac{1}{2V_\infty}\left(1-\frac{1}{\sqrt{1-M_\infty^2}}\right) \tag{19}$$

where $M_\infty = \frac{V_\infty}{C_\infty}$ is the Mach-number.

- C. Iacob approximation variant:

$$C_1 = \frac{1}{2}\left[1+\left(1-\frac{\gamma-1}{2}M_0^2\right)^{-\frac{1}{\gamma}-1}\right] \tag{20}$$

$$C_2 = \frac{\gamma}{2C_0^2}\frac{1-\left(1-\frac{\gamma-1}{2}M_0^2\right)^{-\frac{1}{\gamma-1}}}{\left[1-\left(1-\frac{\gamma-1}{2}M_0^2\right)^{\frac{\gamma}{\gamma-1}}\right]\left[1+\left(1+\frac{\gamma-1}{2}M_0^2\right)^{-\frac{1}{\gamma-1}}\right]}$$

4 J. Leray Transposition and Practical Application

To have a bijective transposition between the (z) and (ζ) planes (to close the specialized plane obstacles, with scaling parameter t and considered angle λ) the relations of Theorem 1 will be changed to be valid in case of circulational motion too. In this way, the (z) compressible plane circulational motion will be equal with an uncompressible, fictitious circulational motion in plane (ζ) (by using the (14), (15), (16), (17) equations). This is known as the J. Leroy transposition:

$$\left(\overline{w}(\zeta) = \frac{1}{h(\zeta)}\frac{df}{d\zeta}\right) \tag{21}$$

For the (ζ) plane's canonical region the uncompressible fluid's flux domain will be chosen around a unit radius circle. In other words, the principal periodical strip's (D_0^-) domain conform projection will be the flux of $|\zeta| = 1$ unit radius circle.

For univocal transposition it is laid down that the principal periodical strip (D_0^-) is equal with the $|\zeta| = 1$ unit radius circle:

Property 7 [11]**:** From the Kutta-Zsukovszki hypothesis validation, the *R* radius circle *Γ* circulation is given by the following equation:

$$\Gamma = \frac{4RtV_1\left(R^2 - 2R\cos\theta + 1\right)}{\left(R^2 - 1\right)\left(R^4 - 2R^2\cos 2\theta_0 + 1\right)} \cdot \begin{bmatrix} R^2 \sin(\alpha_1 - \lambda - \theta_0) - \\ -\sin(\alpha_1 - \lambda - \theta_0) \end{bmatrix} \tag{22}$$

Property 8 [11]: The complex potential of the flux around the uncompressible $|\zeta| = 1$ unit radius circle is given by the following equation:

$$f(\zeta) = \frac{Q - i\Gamma_1}{2\pi} \ln\frac{\zeta + R}{\zeta - R} + \frac{Q + i\Gamma_1}{2\pi} \ln\frac{\zeta + \frac{1}{R}}{\zeta - \frac{1}{R}} - i\frac{\Gamma}{2\pi} \ln\frac{\zeta - R}{\zeta - \frac{1}{R}} \tag{23}$$

where *Q* and *Γ* physical parameters are given by equation (1).

Remark 4: In equations (22) and (23) the value of *R* is determined empirical from a table (according to *λ* considered angle and *t*/*l* density – [11], pp. 113, table 1).

Theorem 2 [5]**:** The shift (slip) of the corresponding blade profile contours of the (*z*) and (ζ) planes according to relation (14) is given by the following equation:

$$\tau = \omega_2 - \omega_1, \quad \omega_1 = \frac{c_1 + c_2\left(v_{1i}e^{i\alpha_1}\right)^2}{c_1^2 + c_2^2 v_{1i}^4} \cdot \omega, \quad \omega_2 = \frac{c_1 + c_2\left(v_{2i}e^{i\alpha_2}\right)^2}{c_1^2 + c_2^2 v_{2i}^4} \cdot \omega \tag{24}$$

Property 9 [5]**:** The J. Leray $h(\zeta)$ function is determined using the hypothesis of closing the blade profiles:

$$\tau = \omega_2 - \omega_1 = 0 \tag{25}$$

Theorem 3 [5]**:** In case of networks with straight discs (as the result of F. Weinig [8]), the J. Leroy-type $h(\zeta)$ function has the following expression:

$$h(\zeta)=\frac{\left(1-\frac{S}{\zeta}\right)\left(1-\frac{1}{\zeta}e^{i\theta_0}\right)^{1-\frac{\varepsilon}{\pi}}}{\left(1+\frac{1}{R\zeta}\right)\left(1-\frac{1}{R\zeta}\right)\left(1-\frac{T}{\zeta}\right)^{2-\frac{\varepsilon}{\pi}}}, \quad \overline{w}(\zeta)=\frac{1}{h(\zeta)}\frac{df}{d\zeta} \tag{26}$$

where

- S is the point on the stagnation point's radius, defined by θ_0 at a 0.0125-0.15 distance from the circle;
- T is the interior point of the basic circle $|\zeta|=1$, defined by the relation:

$$C_1+C_2\left(V_{1i}e^{i\alpha_1}\right)^2=h(-R)\left[C_1^2+C_2^2V_{1i}^4\right] \tag{27}$$

Remark 5: Based on the results presented above a corresponding algorithm for practical application was developed by the authors [3].

It can be confirmed easily that using the method developed in the current paper, the C. C. Lin method [6], can be obtained (if the C_1 and C_2 constants are replaced with the relations of the Kármán-Tschien variant).

Conclusions

The paper presents a generalization of the C. C. Lin variant [6]. In this way, it is capable for given values of the C_1 and C_2 constants to analyze the compressible fluid's flux through profiles' network by using all three possibilities of the known hodographic methods. If the C_1 and C_2 constants are determined by the Kármán-Tschien approximation variant, then the C. C. Lin method is obtained.

The mathematical modeling method can be used in practice even if the network elements are not blade profiles, but the elements are physically and geometrically fulfilling the hypothesis of a periodic system.

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