Some Extensions of Migrativity for Triangular Norms

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Abstract: In this paper we introduce and describe continuous triangular norms that are migrative with respect to another fixed t-norm T_0, in particular to the three prototypes $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$. Depending on characteristic properties of T_0, classes of nilpotent and strict migrative t-norms are naturally formed. In these cases the characterization and construction is carried out by solving functional equations for the generators. In the third case an ordinal-sum-like construction is resulted.

Keywords: triangular norm, migrative property, additive generator, functional equations.

1 Introduction

In [3] the authors introduced the new term – α-migrative – for a class of binary operations as follows.

Definition 1. Let α be in $]0,1[$. A binary operation $T\colon [0,1]^2 \to [0,1]$ is said to be α-*migrative* if we have

$$T(\alpha x, y) = T(x, \alpha y) \quad \text{for all } x, y \in [0,1]. \tag{1}$$

One can easily see that the following function $T_\beta\colon [0,1]^2 \to [0,1]$ is α-migrative (where $\beta \in [0,1]$):

$$T_\beta(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ \beta xy & \text{otherwise.} \end{cases} \tag{2}$$

In fact, thus defined function T_β is a triangular norm for any $\beta \in [0,1]$.

A *triangular norm* (t-norm for short) $T : [0,1]^2 \to [0,1]$ is an associative, commutative, non-decreasing function such that $T(1,x) = x$ for all $x \in [0,1]$. Prototypes of t-norms are the minimum $T_{\mathbf{M}}(x,y) = \min(x,y)$, the product $T_{\mathbf{P}}(x,y) = xy$, and the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. Obviously, the product t-norm $T_{\mathbf{P}}$ is α-migrative for any $\alpha \in]0,1[$.

As it is well-known, each continuous Archimedean t-norm T can be represented by means of a continuous additive generator (see e.g. [6]), i.e., a strictly decreasing continuous function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)), \tag{3}$$

where $t^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of t, and is given by

$$t^{(-1)}(u) = t^{-1}(\min(u, t(0))).$$

A *triangular subnorm* (t-subnorm for short) $T\colon [0,1]^2 \to [0,1]$ is an associative, commutative, non-decreasing function such that $T(x,y) \le \min(x,y)$ for all $x, y \in [0,1]$. Obviously, any t-norm is a t-subnorm. Notice that the function $T'_\beta(x,y) = \beta xy$ for all $x, y \in]$ is a t-subnorm that is also α-migrative for any $\alpha \in \,]0,1[$.

Consider a t-norm $T\colon [0,1]^2 \to [0,1]$. Then T satisfies the associativity functional equation (4), which is well-known in several theoretical and applied fields, and is formulated as follows ($x, y, z \in [0,1]$):

$$T(T(x,y),z) = T(x, T(y,z)). \tag{4}$$

If we fix the value of x, say $x = \alpha$, then equation (4) remains valid for T. Let us choose one particular t-norm T_0, and consider the following functional equation ($x, y \in [0,1]$):

$$T(T_0(\alpha,x),y) = T(x, T_0(\alpha,y)). \tag{5}$$

Then, obviously, T_0 itself is a solution. The question is natural: is there any solution T of (5) that differs from T_0? If so, determine and characterize all solutions.

The *generalized associativity equation* has also been studied and solved, see [1,7]. It can be written as follows:

$$F(G(x,y),z) = H(x, K(y,z)). \tag{6}$$

In this general framework the particular form of $H = F$, $K = G$ in (6) corresponds to (5).

When $T_0 = T_{\mathbf{P}}$, one can recognize α-migrativity (1) as a particular case of (5). The next definition extends the migrative property as follows.

Definition 2. Let α be in $]0,1[$ and T_0 a fixed triangular norm. A binary operation $T\colon [0,1]^2 \to [0,1]$ is said to be *α-migrative with respect to* T_0 (shortly: (α, T_0)-migrative) if we have (5) for all $x, y \in [0,1]$.

Notice that if a t-norm T is (α, T_0)-migrative then we have

$$T(\alpha, y) = T_0(\alpha, y) \quad \text{for all } y \in [0,1]. \tag{7}$$

This follows from (5) by substituting $x = 1$.

In the present paper we study three particular cases of (α, T_0)-migrative t-norms according to the three prototypes. That is, when $T_0 = T_{\mathbf{M}}$, when $T_0 = T_{\mathbf{P}}$, and when $T_0 = T_{\mathbf{L}}$. Notice that the second case was investigated in [4], where all the details and proofs can also be found. The other cases will be published in our forthcoming paper [5].

2 $(\alpha, T_{\mathbf{M}})$-migrative Continuous Triangular Norms

In the present case the $(\alpha, T_{\mathbf{M}})$-migrative property is read as follows:

$$T(\min(\alpha, x), y) = T(x, \min(\alpha, y)) \quad \text{for all } x, y \in [0, 1]. \tag{8}$$

Now (7) implies that $T(\alpha, y) = \min(\alpha, y)$ for all $y \in [0, 1]$.

The description of all $(\alpha, T_{\mathbf{M}})$-migrative continuous triangular norms is given in the following theorem. For the proof see [5].

Theorem 1. *A continuous t-norm T is $(\alpha, T_{\mathbf{M}})$-migrative if and only if there exist two continuous t-norms T_1 and T_2 such that T can be written in the following form:*

$$T(x, y) = \begin{cases} \alpha T_1\left(\dfrac{x}{\alpha}, \dfrac{y}{\alpha}\right) & \textit{if } x, y \in [0, \alpha], \\ \alpha + (1-\alpha) T_2\left(\dfrac{x-\alpha}{1-\alpha}, \dfrac{y-\alpha}{1-\alpha}\right) & \textit{if } x, y \in [\alpha, 1], \\ \min(x, y) & \textit{otherwise}. \end{cases}$$

3 $(\alpha, T_{\mathbf{P}})$-migrative Continuous Triangular Norms

The $(\alpha, T_{\mathbf{P}})$-migrative property now is read as follows:

$$T(\alpha x, y) = T(x, \alpha y) \quad \text{for all } x, y \in [0, 1], \tag{9}$$

This is the original α-migrativity, and (7) implies that $T(\alpha, y) = \alpha y$ for all $y \in [0, 1]$.

We have shown that the migrative property is rather strong for a continuous t-norm: it implies that the t-norm cannot have idempotent elements, and cannot be nilpotent.

Theorem 2. *Let T be a continuous t-norm. If T is α-migrative then T is strict.*

It is easy to conclude (see [3]) that a strict t-norm T with additive generator t is α-migrative if and only if

$$t(\alpha x) - t(x) = t(\alpha y) - t(y) \quad \text{for all } x, y \in [0, 1]. \tag{10}$$

Equation (10) says that the difference $t(\alpha x) - t(x)$ is independent of x. More exactly, if we chose $y = 1$ in (10), this independent difference can be obtained as $t(\alpha x) - t(x) = t(\alpha)$. We write it as follows:

$$t(\alpha x) = t(\alpha) + t(x) \quad \text{for all } x \in [0, 1]. \tag{11}$$

In the next theorem we provide the general solution of the functional equation (11). It is based on the important fact that the restriction of t to the interval $[\alpha, 1]$ uniquely determines t on each subinterval $[\alpha^{k+1}, \alpha^k]$, progressing from left to right.

Theorem 3. *Suppose t is an additive generator of a strict t-norm. Then t satisfies the functional equation (11) if and only if there exists a continuous, strictly decreasing function t_0 from $[\alpha, 1]$ to the non-negative reals with $t_0(0) < +\infty$ and $t_0(1) = 0$ such that*

$$t(x) = k \cdot t_0(\alpha) + t_0\left(\frac{x}{\alpha^k}\right) \quad \textit{if } x \in]\alpha^{k+1}, \alpha^k], \tag{12}$$

where k is any non-negative integer.

Unfortunately, none of the famous t-norm families (like Frank, Hamacher, Dombi, Alsina) are migrative, except the particular case of $t(x) = -\log x$, or equivalently, $T(x, y) = T_{\mathbf{P}}(x, y) = xy$.

This results is illustrated in the next figure with $\alpha = \dfrac{3}{4}$, $t_0(x) = 4 - 4x$ for $x \in \left[\dfrac{3}{4}, 1\right]$. Then $t\left(\left(\dfrac{3}{4}\right)^k\right) = k$, and t is linear in between.

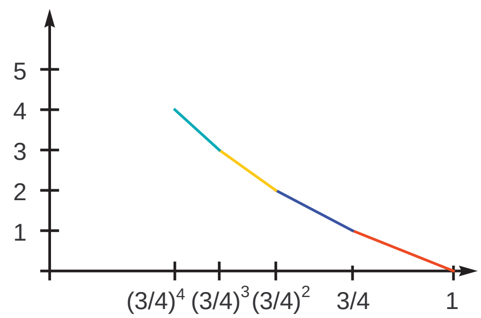

Figure 1
Additive generator of a $3/4$-migrative t-norm

For further results for instance on the construction of smooth additive generators and proofs we refer to [4].

4 $(\alpha, T_{\mathbf{L}})$-migrative Continuous Triangular Norms

In the present case the $(\alpha, T_{\mathbf{L}})$-migrative property is read as follows:

$$T(\max(\alpha + x - 1, 0), y) = T(x, \max(\alpha + y - 1, 0)) \quad \text{for all } x, y \in [0, 1]. \tag{13}$$

Now (7) implies that $T(\alpha, y) = \max(\alpha + y - 1, 0)$ for all $y \in [0, 1]$.

The description of all $(\alpha, T_{\mathbf{L}})$-migrative continuous triangular norms is given now. For proofs and more details see [5].

Lemma 1. *Assume that T is a continuous t-norm that is $(\alpha, T_{\mathbf{L}})$-migrative. Then there exists an automorphism φ of the unit interval such that $T = T_{\mathbf{L}}^{\varphi}$. That is, we have*

$$T(x, y) = T_{\mathbf{L}}^{\varphi}(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0)) \quad \textit{for all } x, y \in [0, 1]. \tag{14}$$

Taking into account the functional form of T given in (14), the equation (12) defining $(\alpha, T_{\mathbf{L}})$-migrativity has the following form:

$$\varphi^{-1}(\max[\varphi(\max(\alpha+x-1,0))+\varphi(y)-1,0]) = \\ = \varphi^{-1}(\max[\varphi(x)+\varphi(\max(\alpha+y-1,0))-1,0]).$$

If we apply φ to both sides of this equality we get the following equivalent form of (14) ($x, y \in [0,1]$):

$$\max[\varphi(\max(\alpha+x-1,0))+\varphi(y)-1,0] = \\ = \max[\varphi(x)+\varphi(\max(\alpha+y-1,0))-1,0]. \quad (15)$$

This equation implies that

$$\varphi(\max(\alpha+x-1,0))+\varphi(y)>1 \iff \varphi(x)+\varphi(\max(\alpha+y-1,0))>1.$$

In particular, it is absolutely necessary for having these strict inequalities that $\alpha + x > 1$ and $\alpha + y > 1$. In this case we can write

$$\alpha+x-1>\varphi^{-1}(1-\varphi(y)) \iff \alpha+y-1>\varphi^{-1}(1-\varphi(x)), \quad (16)$$

and for such x, y the automorphism φ must satisfy the following functional equation:

$$\varphi(\alpha+x-1)+\varphi(y)=\varphi(x)+\varphi(\alpha+y-1). \quad (17)$$

As a consequence of (16) and (17) we get (by choosing $y = 1$) that

$$\alpha > 1-x \iff \alpha > \varphi^{-1}(1-\varphi(x)) \quad (18)$$

and

$$\varphi(\alpha+x-1)=\varphi(\alpha)+\varphi(x)-1. \quad (19)$$

In addition, continuity of φ implies also that $\alpha = 1 - x$ if and only if $\alpha = \varphi^{-1}(1-\varphi(x))$. That is,

$$\varphi(\alpha)+\varphi(1-\alpha)=1. \quad (20)$$

If we take into account (20) in (19) we get

$$\varphi(x-(1-\alpha))=\varphi(x)-\varphi(1-\alpha). \quad (21)$$

That is, if we know φ on the interval $[1-\alpha, 1]$ then equation (21) defines φ on $[0, \alpha]$.

Theorem 4. *Assume that $\alpha < 1/2$. A t-norm $T(x,y) = \varphi^{-1}(\max(\varphi(x)+\varphi(y)-1,0))$ is $(\alpha, T_{\mathbf{L}})$-migrative if and only if there exist automorphisms ψ_0 and ψ_1 of the unit interval and a real number $0 < \gamma < 1/2$ such that*

$$\varphi(x) = \begin{cases} \gamma\psi_0\left(\dfrac{x}{\alpha}\right) & \text{if } 0 \le x \le \alpha, \\ (1-2\gamma)\psi_1\left(\dfrac{x-\alpha}{1-2\alpha}\right)+\gamma & \text{if } \alpha < x < 1-\alpha, \\ \gamma\psi_0\left(\dfrac{x-(1-\alpha)}{\alpha}\right)+1-\gamma & \text{if } 1-\alpha \le x \le 1. \end{cases} . \quad (22)$$

Complementary to this result, we have to consider the case when $\alpha \geq 1/2$ – that is, when $\alpha \geq 1-\alpha$. We start from an arbitrary automorphism ψ_0 of the unit interval, a number $\gamma \in]0,1[$, and define a piece of the automorphism φ in (15) as follows:

$$\varphi(x) = \gamma \cdot \psi_0 \left(\frac{x-\alpha}{1-\alpha} \right) + 1 - \gamma, \quad x \in [\alpha, 1]. \tag{23}$$

We have that $\varphi(\alpha) = 1-\gamma$.

Denote by n the largest positive integer k such that $k\alpha - (k-1) > 0$. We can extend the definition of φ from $[\alpha, 1]$ to the intervals $[2\alpha - 1, \alpha]$, ..., $[n\alpha - (n-1), (n-1)\alpha - (n-2)]$. It can be seen that for any $k = 1, \ldots, n$ we have

$$\varphi(k\alpha - (k-1)) = k\varphi(\alpha) - (k-1).$$

To have a meaningful extension, the following inequalities must hold:

$$\frac{n-1}{n} \leq \varphi(\alpha) \leq \frac{n}{n+1}$$

and

$$\frac{n-1}{n} \leq \alpha \leq \frac{n}{n+1}.$$

Then we can define φ for $x \in [k\alpha - (k-1), (k-1)\alpha - (k-2)]$ as follows ($k = 1, \ldots, n$):

$$\varphi(x) = \gamma \cdot \psi_0 \left(\frac{x + (k-1) - k\alpha}{1-\alpha} \right) + 1 - k\gamma, \tag{24}$$

where γ depends on ψ_0 and α as follows:

$$\gamma = \frac{1}{n + 1 - \psi_0 \left(\frac{n - (n+1)\alpha}{1-\alpha} \right)}.$$

This choice of γ guarantees that the definition of φ on $[n\alpha - (n-1), 1]$ is appropriate. This makes it possible that φ can be defined in a meaningful way also on the missing part $[0, n\alpha - (n-1)]$ by equation (19).

All the details of handling this case can be found in [5].

5 Summary and Conclusions

In this paper we have completely described continuous t-norms that are migrative with respect to a fixed t-norm from the prototypes. Their characterization has been developed through solutions of a functional equation.

Although Definition 1 is seemingly general, notice that it does not provide a meaningful notion for triangular conorms. Indeed, if S is a t-conorm then it is α-migrative if and only if $S(\alpha x, y) = S(x, \alpha y)$ holds for all $x, y \in [0,1]$. If we choose $y = 0$ then we must have $\alpha x = x$ for all $x \in [0,1]$, because S is α-migrative. This is impossible when $\alpha \neq 1$. Similarly, if $y = 1$ then we must have $S(x, \alpha) = 1$ for all $x \in [0,1]$, which is again impossible unless $\alpha = 1$. Therefore, even the correct definition of α-migrative t-conorms needs special care.

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