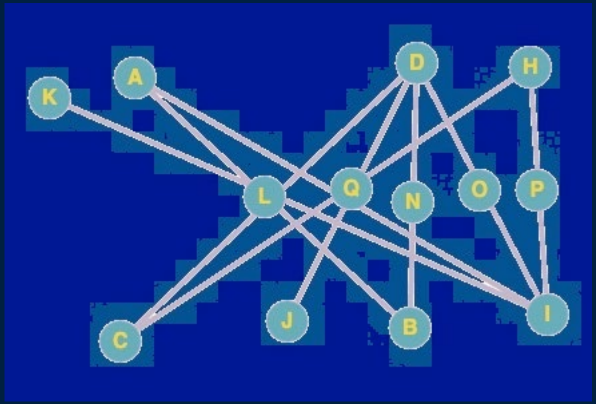
2023 IEEE 23rd International Symposium on Computational Intelligence and Informatics (CINTI)

GRAPHS AND NETWORKS

Theory and Applications

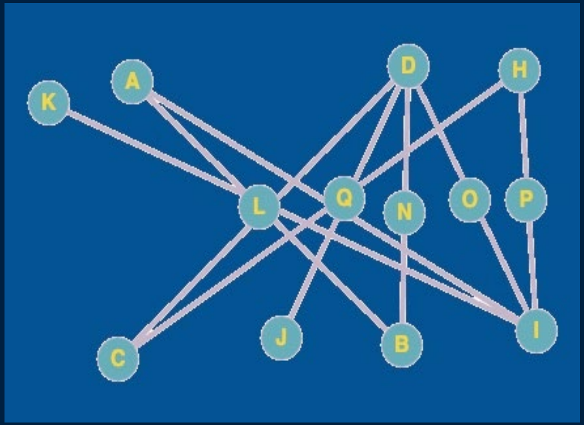
PROF. DR. JOZSEF SOLYMOSI

UNIVERSITY OF BRITISH COLUMBIA, CANADA
OBUDA UNIVERSITY, BUDAPEST, HUNGARY

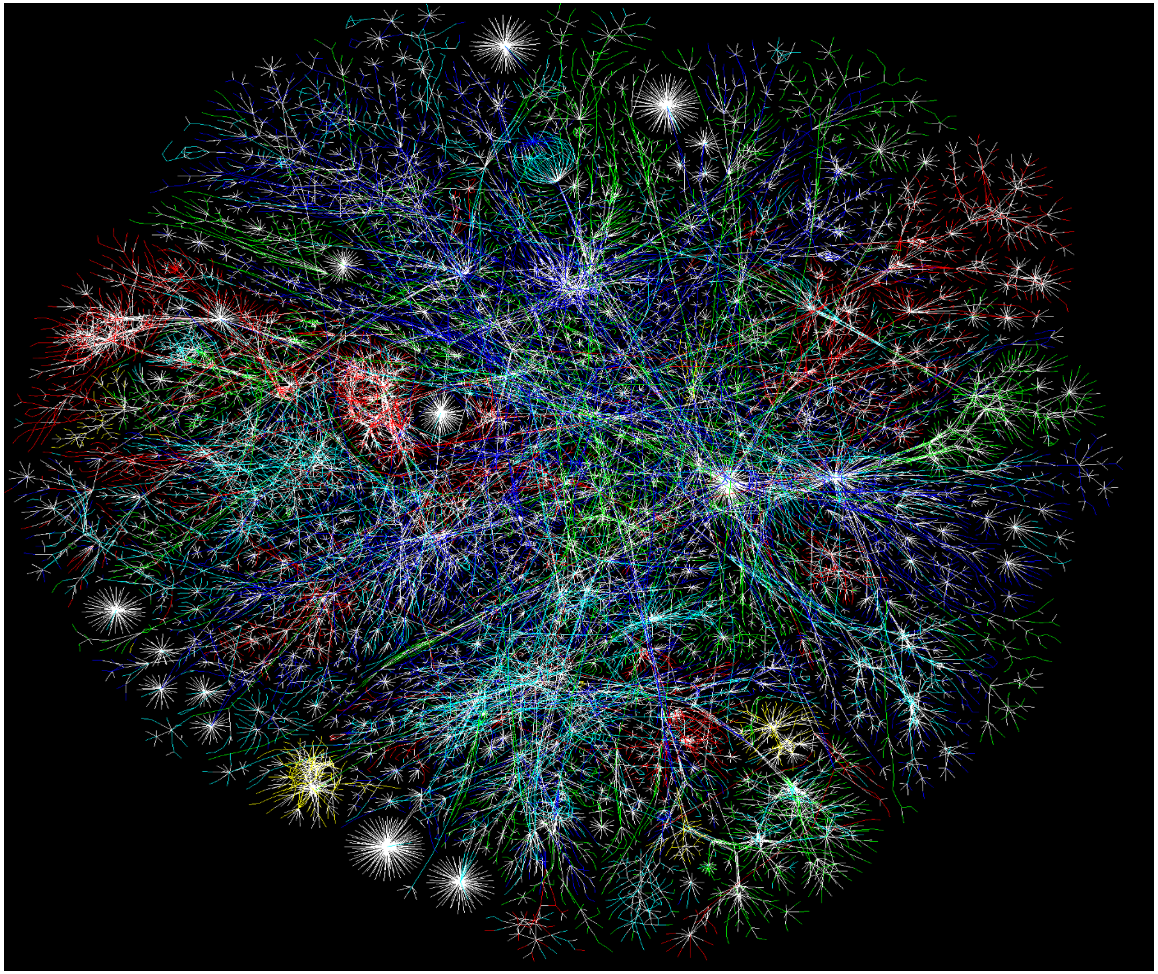


SCHEDULE

- **Graphs are everywhere** (a short intro)
- **Two examples for direct applications:** Chromatic number and Longest path
- **Geometric graphs:**
  - Rigidity of joint-bar frameworks. The non-generic case
  - The unit distance graph
  - Graph drawing. The crossing number

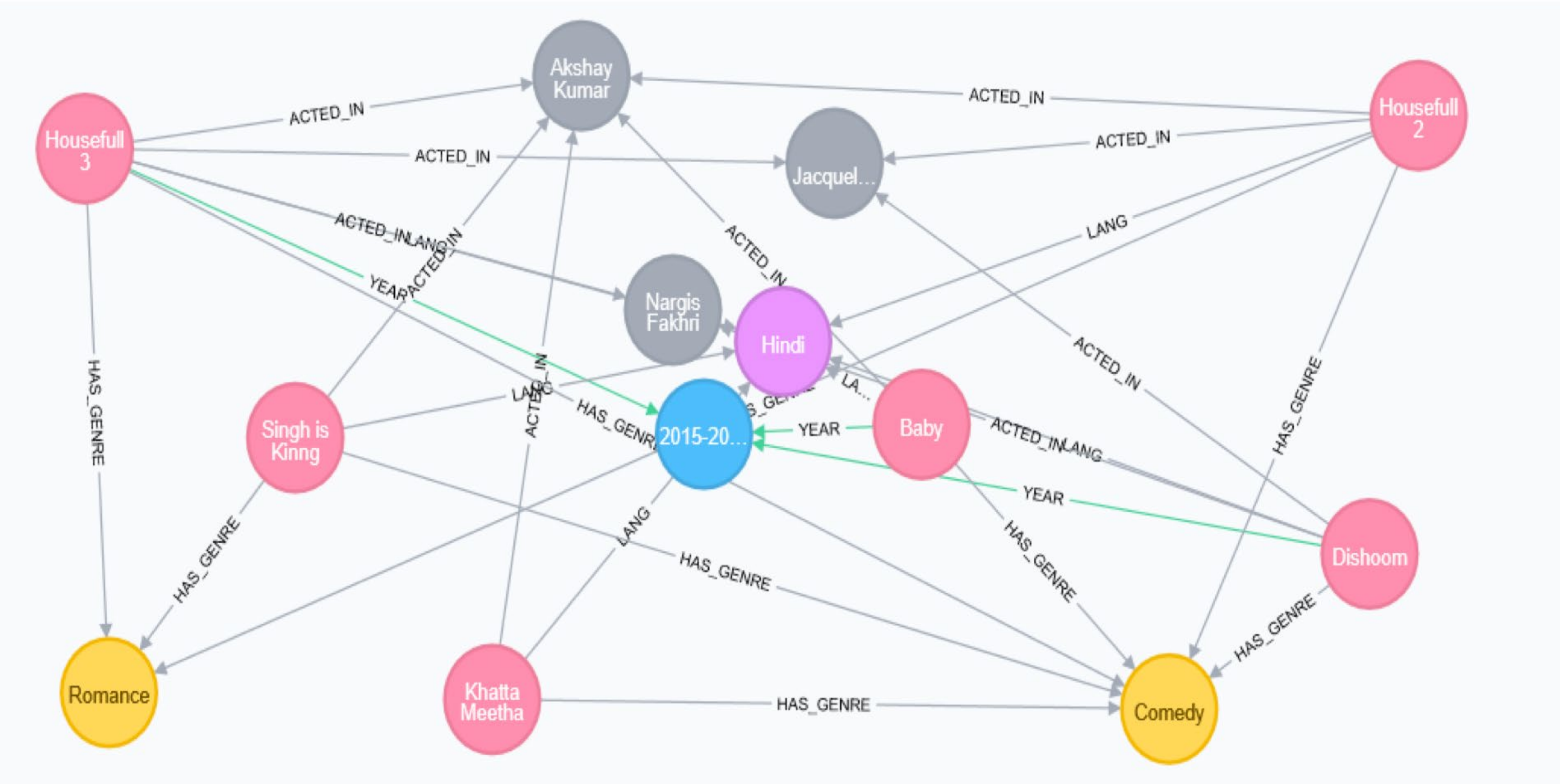


GRAPHS ARE EVERYWHERE



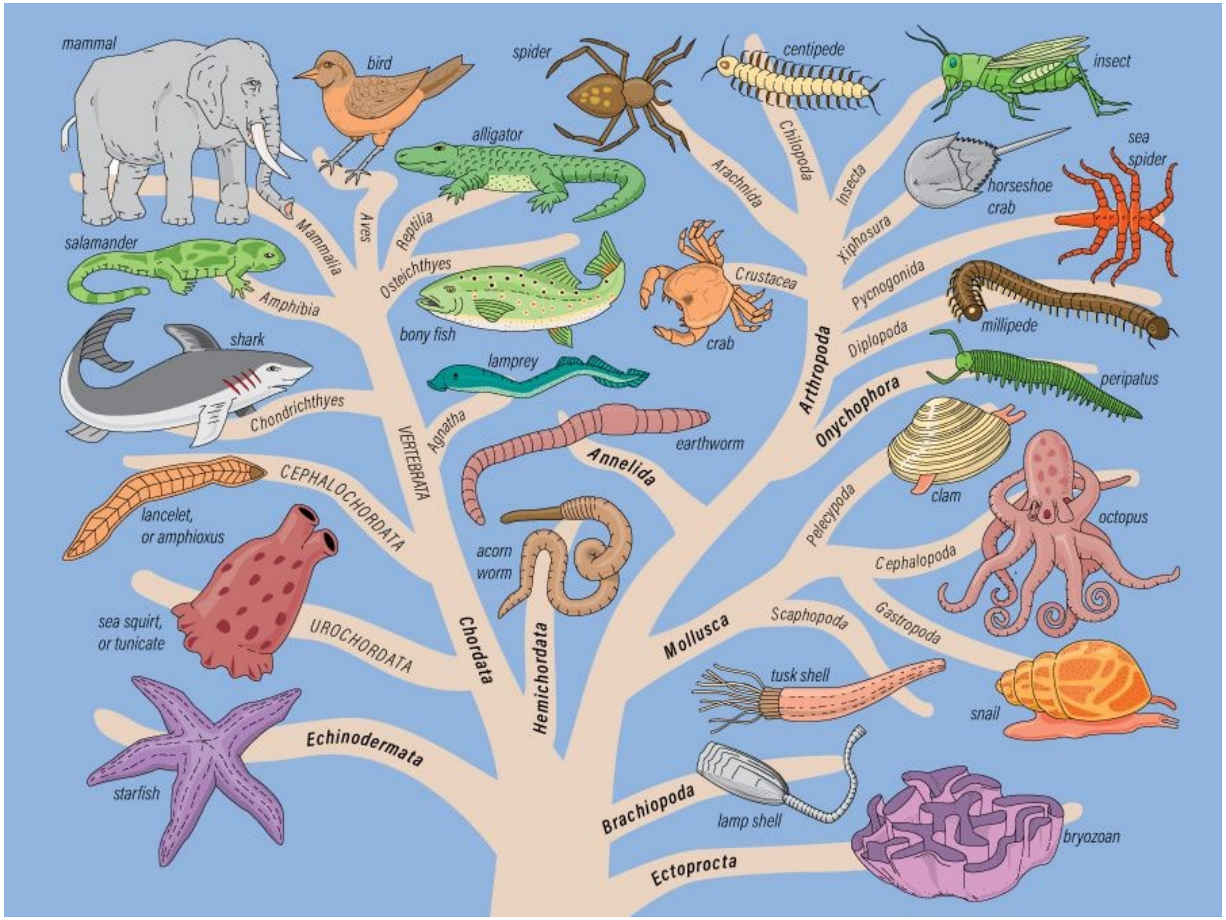
1-day Internet map (23 Nov. 2008):
Red: Asia/Pacifica;
Green, Europe/Middle East/Central Asia/Africa,
Blue: N. America;
Yellow: Latin American and Caribbean,
Cyan: Private Networks;
White: unknown
(www.opte.org/maps/tests/).

GRAPHS ARE EVERYWHERE



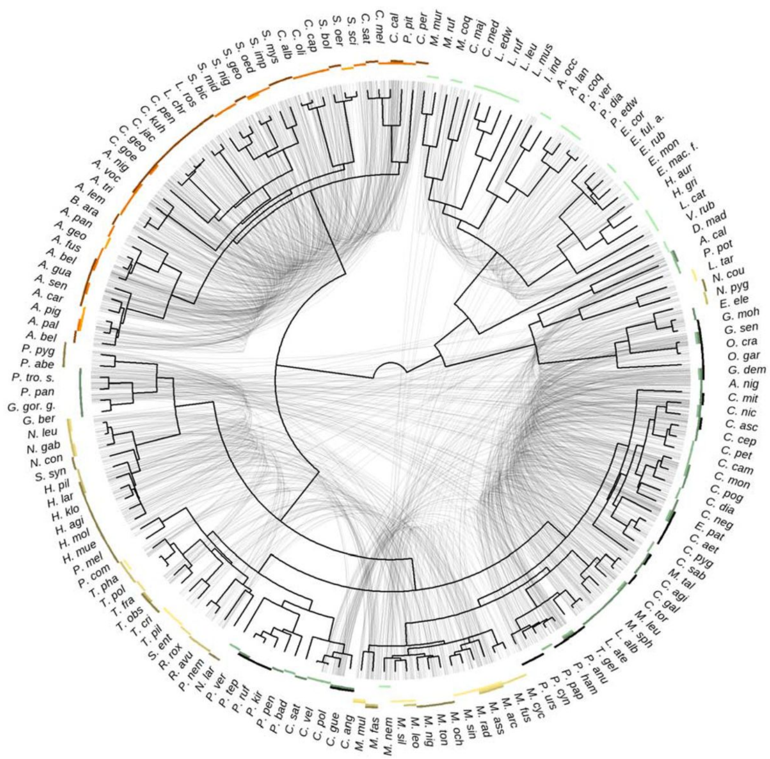
towardsai.net

GRAPHS ARE EVERYWHERE



Dubey, Anna. "How Do You Read Phylogenetic Trees?". *Encyclopedia Britannica*, 27 Jul. 2022

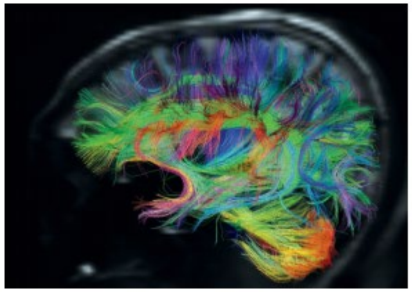
GRAPHS ARE EVERYWHERE



Phylogenetic models of trait evolution: does species sympatry shape brain size evolution?

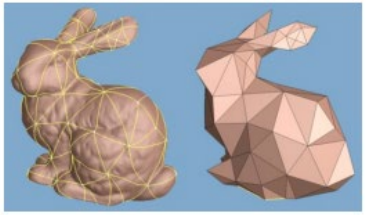
Benjamin Robira, Benoît Perez-Lamarque, Peer Community Journal, Volume 3 (2023), article no. e37.

GRAPHS ARE EVERYWHERE

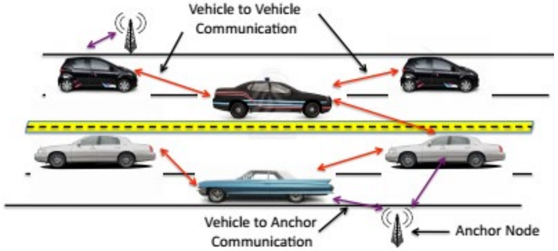


Neuronal Networks

Social Networks



Computer Graphics



Vehicular Networks

Dubey, Anna. "How Do You Read Phylogenetic Trees?". *Encyclopedia Britannica*, 27 Jul. 2022

WHERE GRAPH THEORY APPLIES DIRECTLY

Finding the shortest path in a graph (network) is an easy to solve problem (e.g. Dijkstra's algorithm).

Finding the **longest** path is hard, it is an NP-hard problem.

Even the problem, deciding whether a path exists of at least some given length is NP-complete.

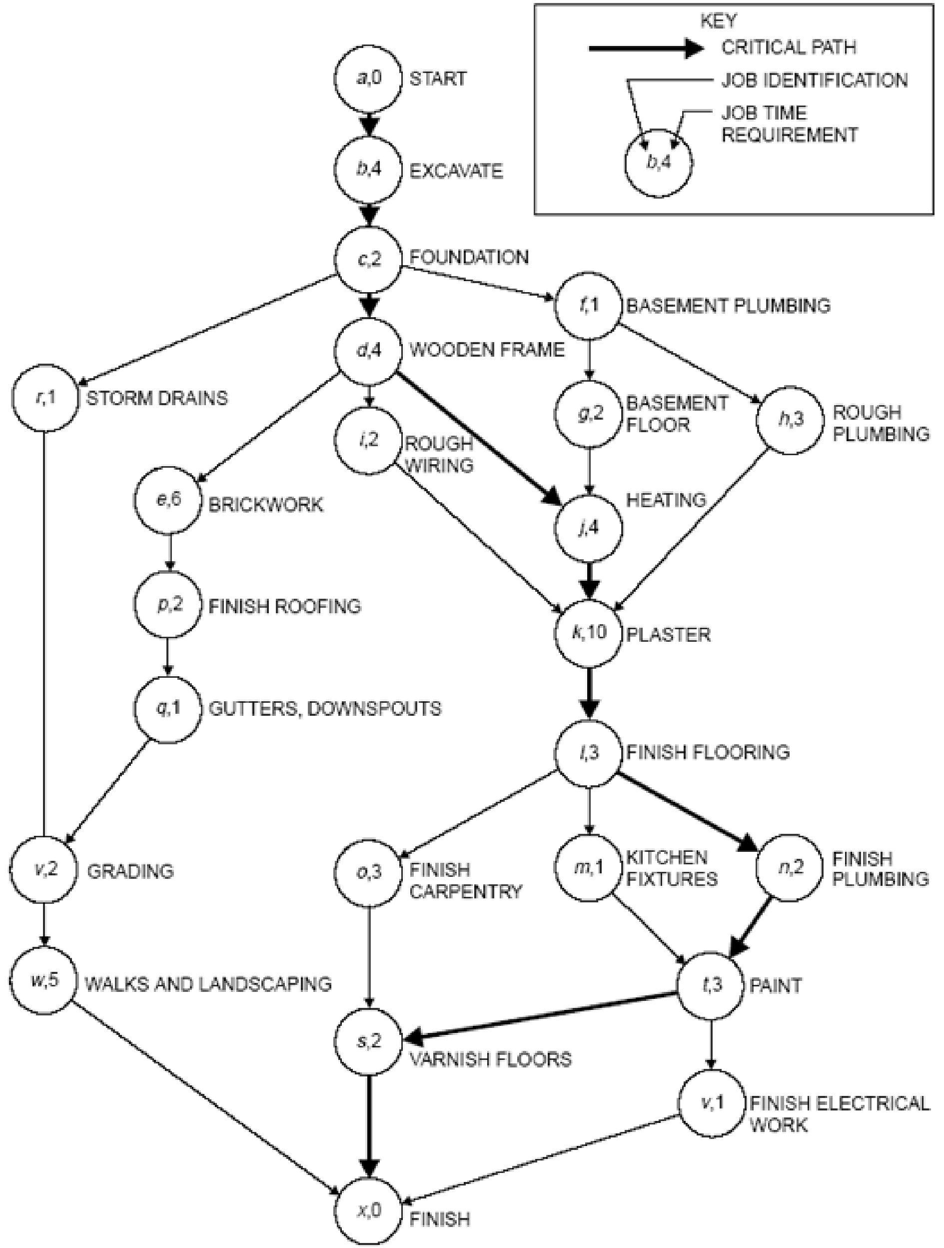
WHERE GRAPH THEORY APPLIES DIRECTLY

A typical application of the longest path algorithm is in scheduling.

Let’s consider a building construction process from start to finish:

digging the foundation, building the walls, installing utilities (water, electricity and gas), doing the interiors, landscaping, etc.

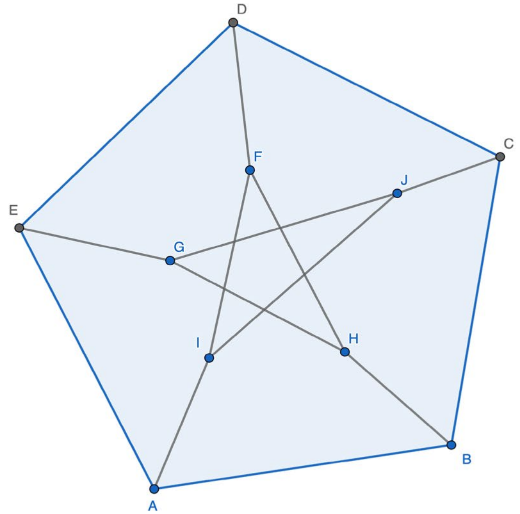
EXHIBIT II
Project Graph



F. K. Levy, G. L. Thompson, and J. D. Wiest,
The ABCs of the Critical Path Method,
Harvard Business Review

WHERE GRAPH THEORY APPLIES DIRECTLY

The chromatic number of a graph is the smallest number of colours needed to colour the vertices such that every edge received different colours in the end-vertices.



The Petersen Graph

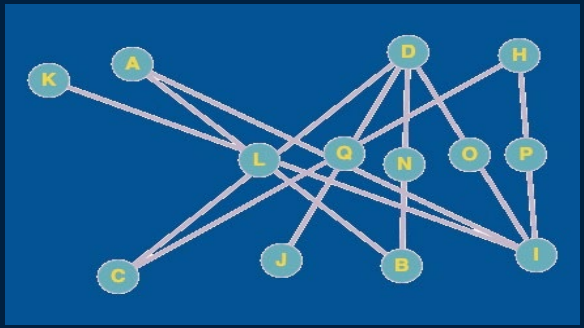
WHERE GRAPH THEORY APPLIES DIRECTLY

Deciding if a graph is 3-colourable
(has chromatic number three)
is an NP-complete problem.

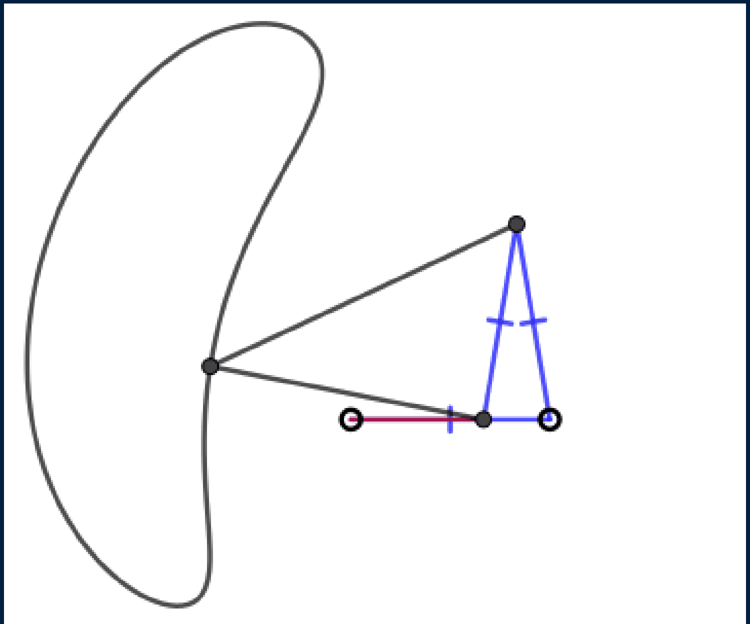
Scheduling the finals at the end of semester is a challenging task. What is the minimum time slots required for the least number of conflicts? The vertices are the courses with finals and two are connected by an edge if there are many students registered to both classes.

GRAPH RIGIDITY AND GRAPH DRAWINGS

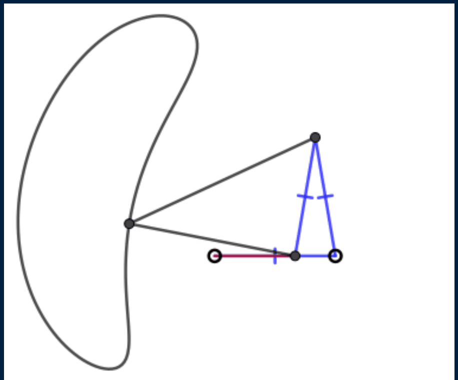
- **Geometric graphs:**
  - Rigidity of joint-bar frameworks. The non-generic case
  - The unit distance graph
  - Graph drawing. The crossing number



BAR-JOINT FRAMEWORKS



BAR-JOINT FRAMEWORKS



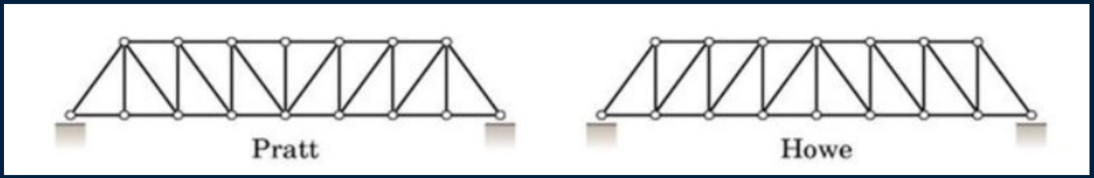
Linkages appear in

- Mechanics
- Biology
- Chemistry
- Kinematics
- +

When can we guarantee that a framework (graph) is rigid, or has a rigid part?

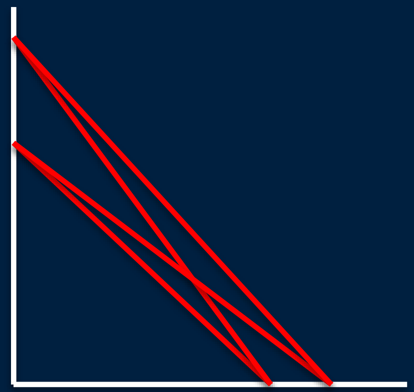
When can we guarantee that a framework (graph) is rigid, or has a rigid part?

If the number of edges of an n-vertex graph is at least 2n-3 and the coordinates of the vertices are algebraically independent, the Laman's theorem guarantees the existence of a rigid part.
Most results in rigidity theory of graphs require some kind of genericity.

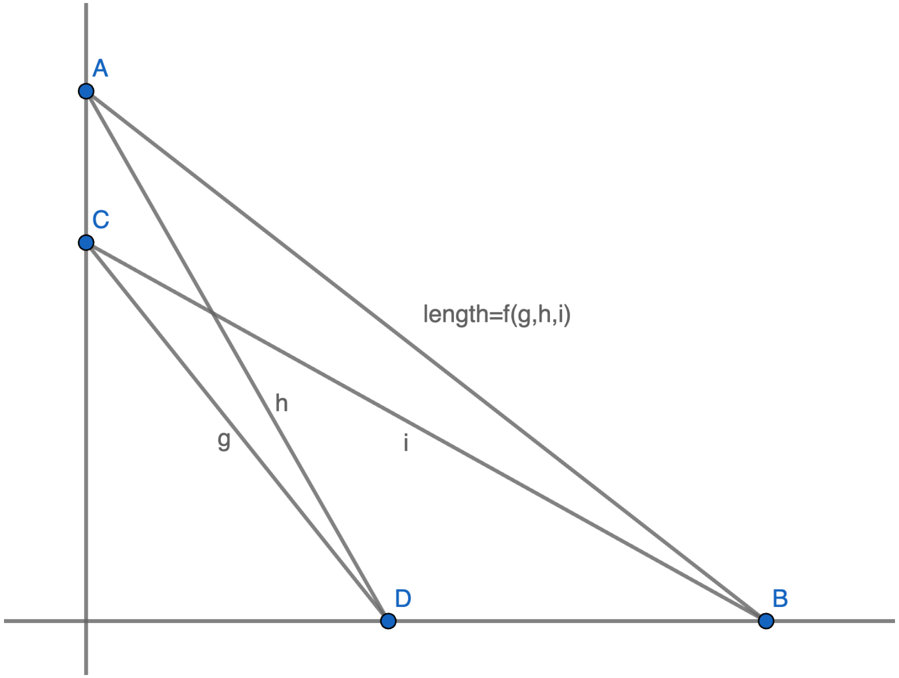


When can we guarantee that a framework (graph) is rigid, or has a rigid part?

Adding more edges (bars) will probably increase rigidity. But this is not always the case.

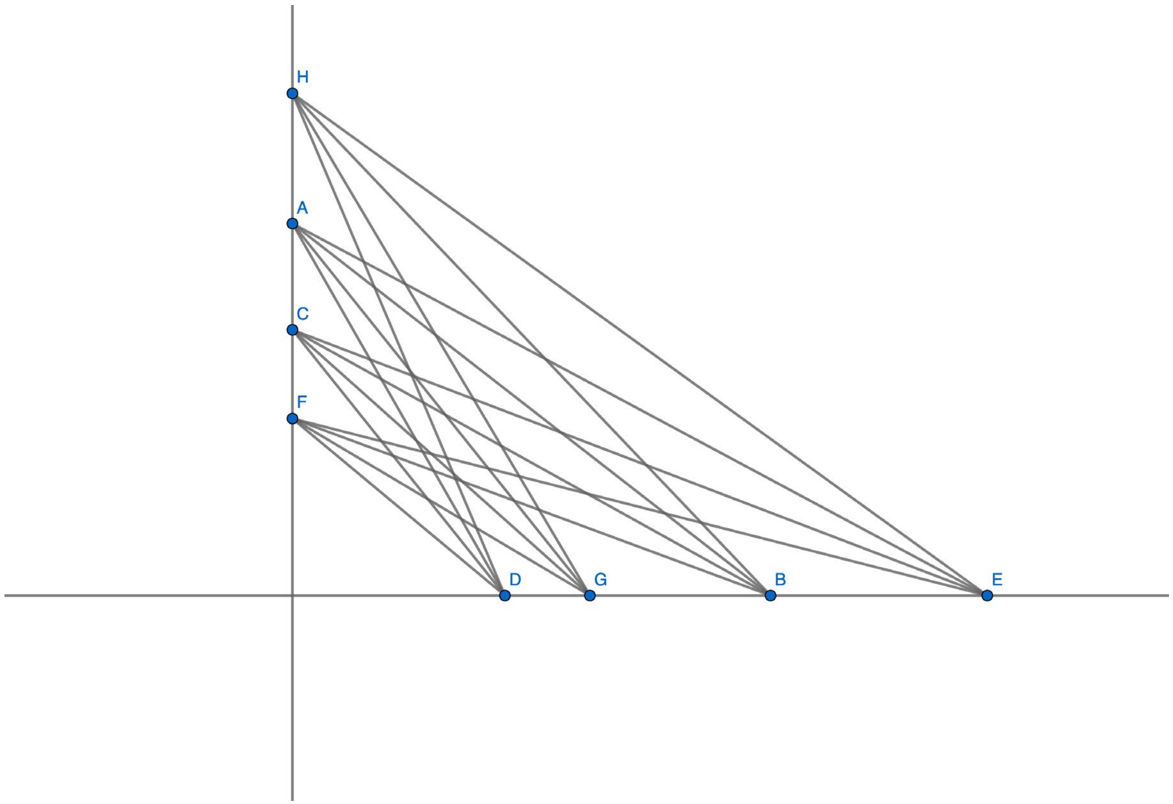


MANY EDGES WITHOUT RIGID PARTS

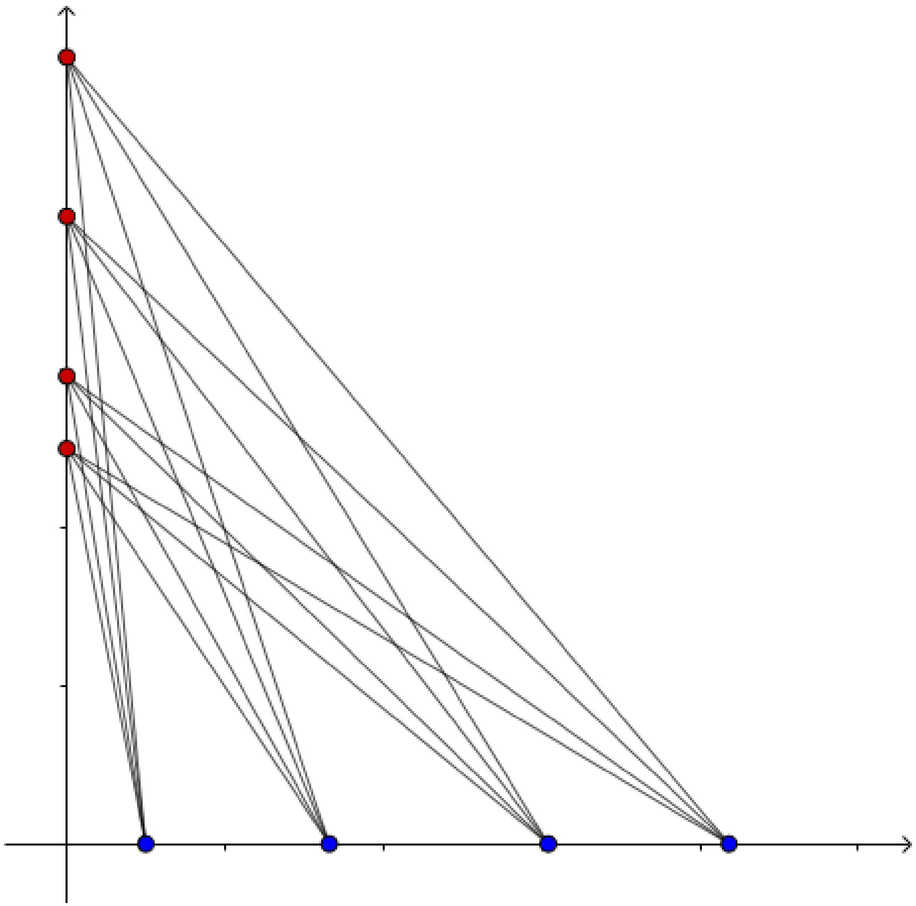


$$\text{length}^2 = h^2 + i^2 - g^2$$

MANY EDGES WITHOUT RIGID PARTS



MANY EDGES WITHOUT RIGID PARTS



Theorem: If a bar-joint framework has n joints and more than $n^{3/2}$ bars then it has rigid parts unless many points are collinear.

Raz, O. E., & Solymosi, J. (2023). Dense Graphs Have Rigid Parts. In Discrete & Computational Geometry. https://doi.org/10.1007/S00454-022-00477-7

The proof of the theorem is quite involved. We are using bounds on point-line incidences in 3-space.

The proof of the theorem is quite involved. We are using bounds on point-line incidences in 3-space.

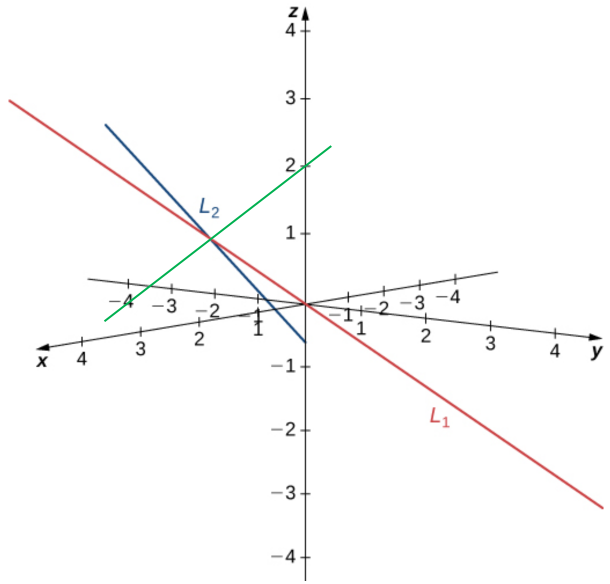
Doubly ruled surfaces:

- the plane
- hyperbolic paraboloid
- single-sheeted hyperboloid

If an arrangement of lines in 3D have many intersections then many of them are from a doubly ruled surface.

The proof of the theorem is quite involved. We are using bounds on point-line incidences in 3-space. Let's see a "lighter" variant of the problem.

Given an arrangement of lines in the 3-space. Three lines define a ***joint*** if they meet in a single point and the lines are not coplanar.



What is the maximum number of joints formed by *n* lines in 3D?

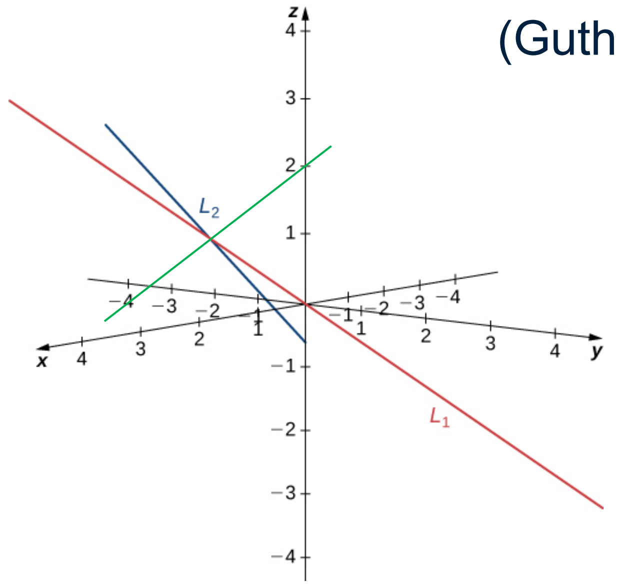
What is the maximum number of joints formed by *n* lines in 3D?

Related works:

Solymosi, J., & Toth, Cs. (2008). On a Question of Bourgain about Geometric Incidences. Combinatorics, Probability and Computing, 17(4), 619-625.

Guth, L., Katz, N.H., Algebraic methods in discrete analogs of the Kakeya problem, *Advances in Mathematics*, Volume 225, Issue 5, 2010, 2828-2839.

Theorem: Any arrangement of *n* lines defines at most $cn^{3/2}$ joints.
(Guth and Katz)



Theorem: Any arrangement of n lines defines at most $cn^{3/2}$ joints.

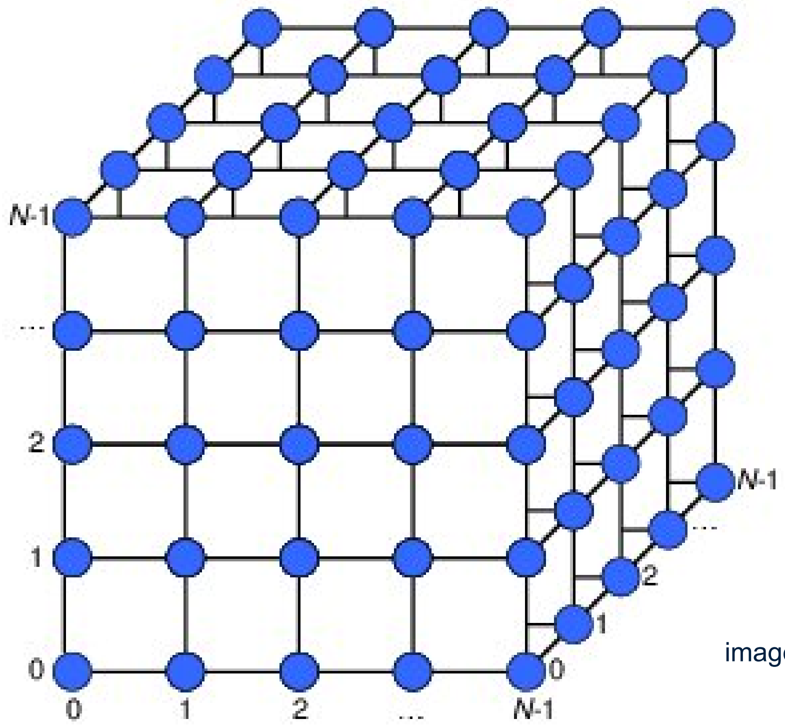


image: José E. Moreira

Example: $3N^2$ lines determine N^3 joints

Theorem: Any arrangement of n lines defines at most $cn^{3/2}$ joints.

To reach a contradiction let us suppose that every line is incident to more than $n^{1/2}$ joints.

Theorem: Any arrangement of n lines defines at most $cn^{3/2}$ joints.

To reach a contradiction let us suppose that every line is incident to more than $n^{1/2}$ joints.

Find an algebraic surface with minimal degree containing all n lines in its surface.

$$f(x,y,z)=0$$

Theorem: Any arrangement of n lines defines at most $cn^{3/2}$ joints.

To reach a contradiction let us suppose that every line is incident to more than $n^{1/2}$ joints.

Find an algebraic surface with minimal degree containing all n lines in its surface.

$$f(x,y,z)=0$$

A simple parameter counting shows that the degree of f is at most $n^{1/2}$.

Theorem: Any arrangement of n lines defines at most $cn^{3/2}$ joints.

To reach a contradiction let us suppose that every line is incident to more than $n^{1/2}$ joints.

Find an algebraic surface with minimal degree containing all n lines in its surface.

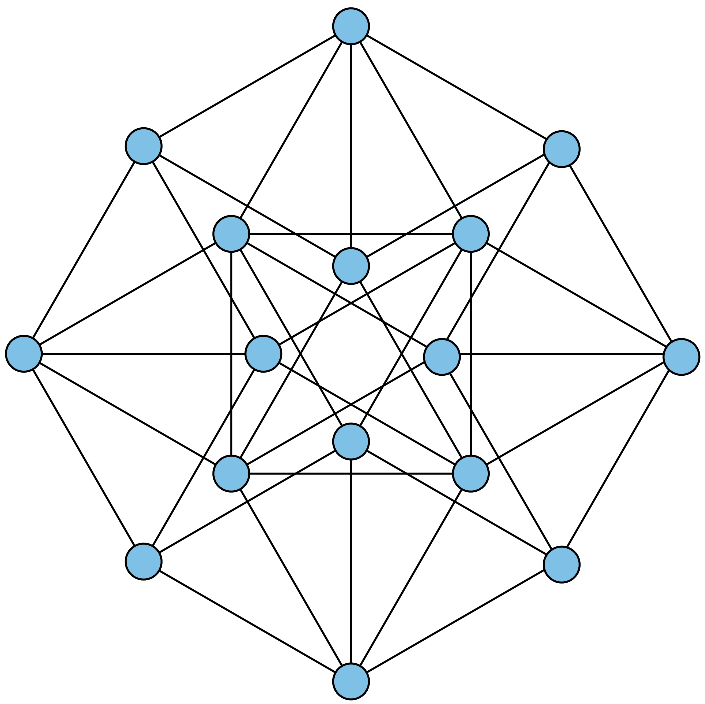
$$f(x,y,z)=0$$

A simple parameter counting shows that the degree of f is at most $n^{1/2}$.

Every joint is a singularity of the surface, the gradient is zero. If a line contributes to more than $n^{1/2}$ joints then it is also a line of the surface defined by the *gradient=0* equation. This equation has a lower degree and still contains all lines.

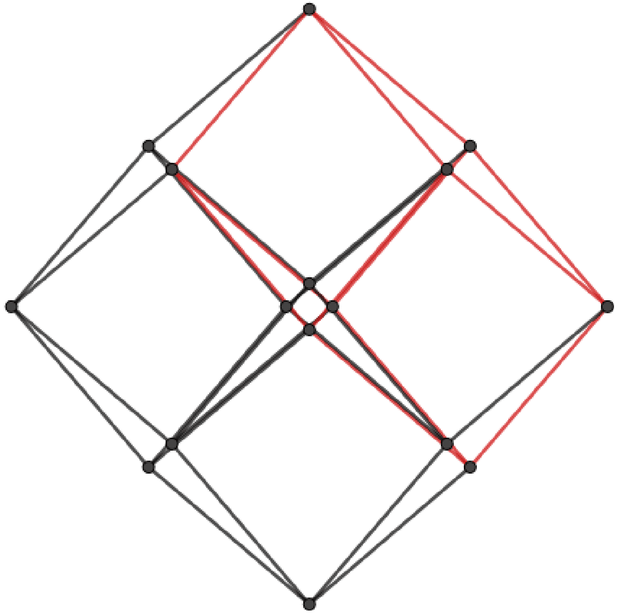
Contradiction.

UNIT DISTANCE GRAPHS



Picture by David Eppstein

UNIT DISTANCE GRAPHS



There are non-rigid frameworks with *nlog(n)* bars

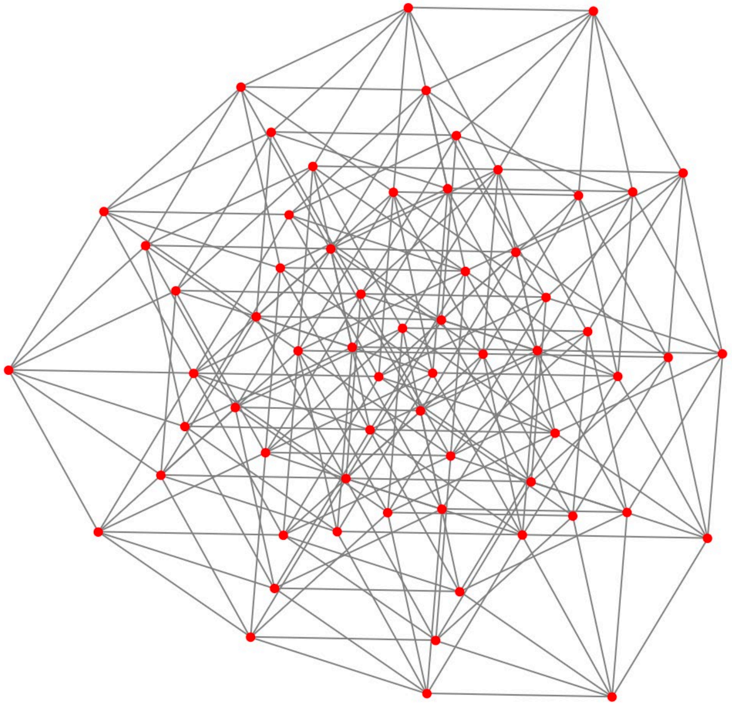
UNIT DISTANCE GRAPHS

There are many open problems related to unit distance graphs. Some of them are more than 50 years old.

- What is the maximum number of edges in a unit distance graph on *n* vertices? (Paul Erdős)
- What is the max chromatic number of a unit distance graph? (What is the chromatic number of the plane? 5-6-7?)

UNIT DISTANCE GRAPHS

What is the maximum number of edges in a unit distance graph on *n* vertices? (Paul Erdős)

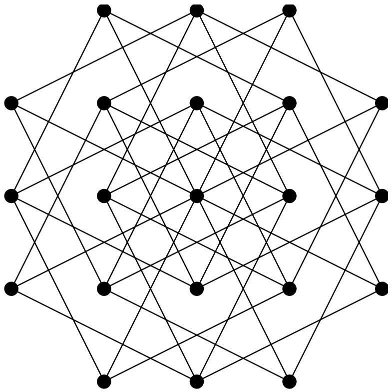


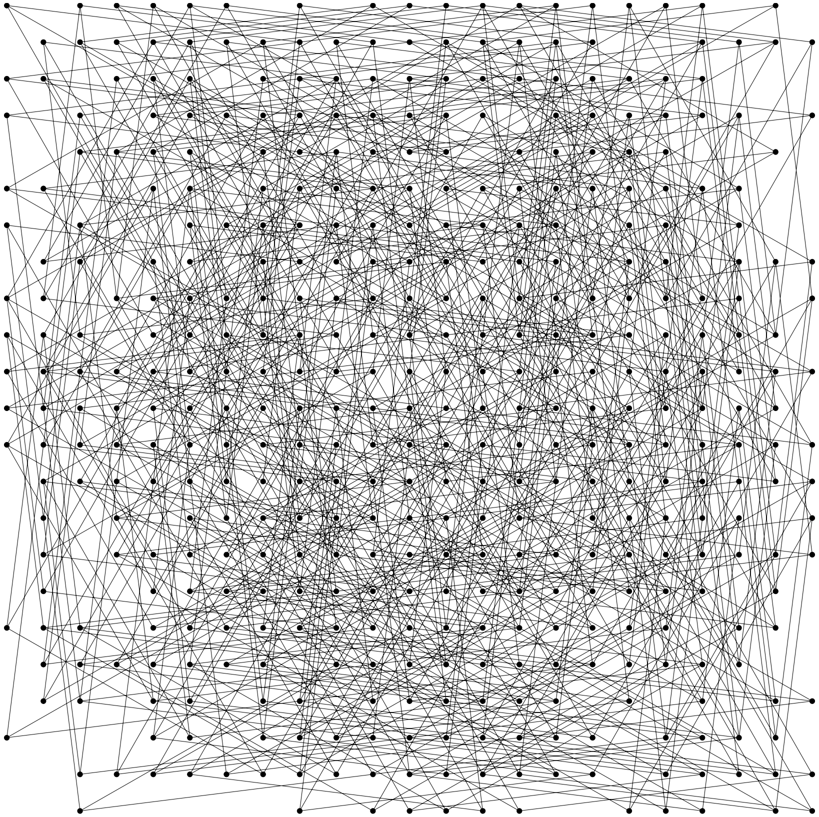
UNIT DISTANCE GRAPHS

Hiroshi Maehara asked if there is a bipartite unit distance graph which is rigid.

- *Maehara, H., & Chinen, K. (1995). An infinitesimally rigid unit-bar-framework in the plane which contains no triangle. Ryuku Mathematical Journal, 8, 37-41.*
- *Maehara, H. (1991). A rigid unit-bar-framework without triangle. Mathematica Japonica, 36, 681-683.*
- *Maehara, H. (2004). Distance graphs and rigidity. Contemporary Mathematics, 342, 149-168.*

UNIT DISTANCE GRAPHS





Solymosi, J., White, E. On Rigidity of Unit-Bar Frameworks.
Graphs and Combinatorics **35**, 1147–1152 (2019).

THE CROSSING NUMBER

The crossing number, *cr(G)*, of a graph *G* is the least number of crossing points in any drawing of *G* in the plane.

THE CROSSING NUMBER

In the design of very large scale integration (VLSI) fabrication technology an interesting theoretical problem aroused: How to minimize the wire crossings in the layout?

THE CROSSING NUMBER

The crossing number, *cr(G)*, of a graph *G* is the least number of crossing points in any drawing of *G* in the plane.

From Euler's formula, *V-E+F=2,* we know that graphs on *n* vertices and *3n-5* edges have at least one crossing in any drawing in the plane. Then graphs with *4n* edges have more than *n* crossings in any drawing.
One can use a probabilistic boosting argument to show that graphs with *e>4n* edges have crossing number at least

$$ce^3/n^2$$

THE CROSSING NUMBER

The previous crossing number bound,

$$cr(G) > ce^3/n^2$$

is sharp up to the constant multiplier, however there are still many open problems.

It is known that the bound can be improved when the degree sequence is uneven, but the exact numbers are not known even for special graphs, like the complete graph or the complete bipartite graph.

Pach, J., Solymosi, J. and Tardos, G. (2010), Crossing numbers of imbalanced graphs. J. Graph Theory, 64: 12-21.

THE UNIVERSITY OF BRITISH COLUMBIA

ÓBUDAI EGYETEM
ÓBUDA UNIVERSITY

Thank You!

IEEE 22nd International Symposium on Computational Intelligence and Informatics (CINTI 2022)