The effect of curvature in isoperimetric problems

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6 September 2022

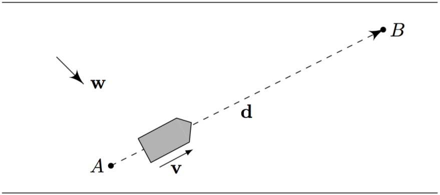
Structure of the talk

- Historical motivations & Own scientific contributions
- Planned research & Research team
- Potential impact of the research

My research area: *Calculus of Variations & Geometric Analysis* (=minimization/maximization of certain functionals on curved spaces).

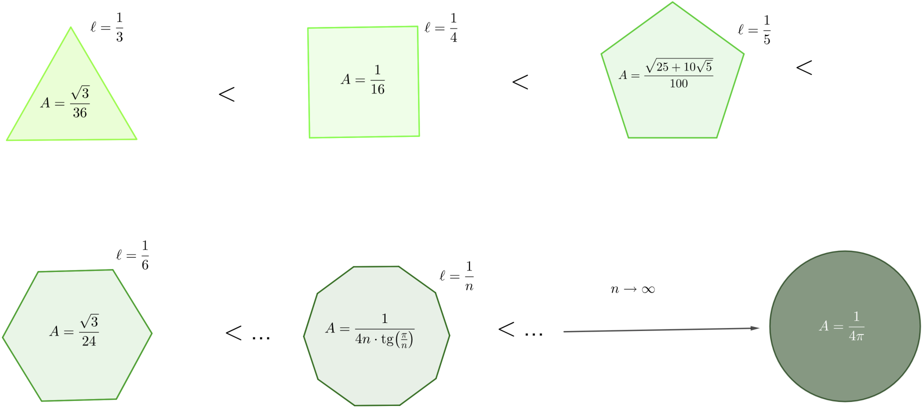
Some topics:

- **Minimizing 'distances'**: to find the 'shortest path/time' between two distinct points (e.g. Zermelo's navigation problem).



- **Partial Differential Equations**: smooth and nonsmooth (e.g. Schrödinger equations, Dirichlet problems, von Kármán adhesive plates, clamped plates, etc).
- **Isoperimetric problems**: to find the shape of a domain of maximal area bounded by a curve of given length.

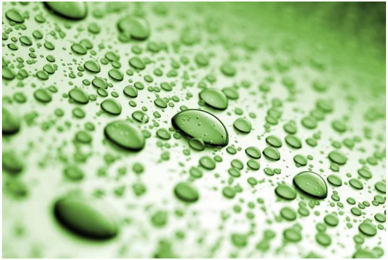
Given a closed yarn/rope of length $L = 1$: what is the domain of maximal area enclosed by it?



Remark: $a_n = \frac{1}{4n\mathrm{tg}(\frac{\pi}{n})}$ is increasing, converging to $\frac{1}{4\pi}$ as $n \to \infty$.

- **Geometry/Architecture** (Dido): maximal area with a given length of wall.

- **Physics** (Drop, soap bubble): minimal surface tension.



Common features: circular/spherical shapes!

Isoperimetric problem:
to find the shape of a domain of maximal area bounded by a curve of given length.

Ancient Greek: ísos = equal.

Splitting of the scientific part:

- Isoperimetric problems in Geometry
- Isoperimetric problems in Physics

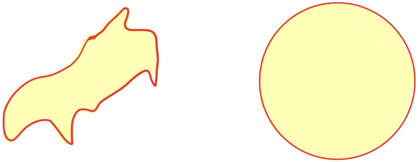
- $n \geq 2$: dimension of the (Euclidean) space.
- $\mathrm{Vol}(\Omega)$: volume of $\Omega \subset \mathbb{R}^n$.
- $\mathcal{P}(\partial\Omega)$: perimeter of Ω.
- ω_n: volume of the n-dimensional unit ball in $\mathbb{R}^n$. (e.g. $\omega_2 = \pi$)

Isoperimetric inequality (in $\mathbb{R}^n$) : *for every bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary, we have*

$$\mathcal{P}(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathrm{Vol}(\Omega)^{\frac{n-1}{n}},$$

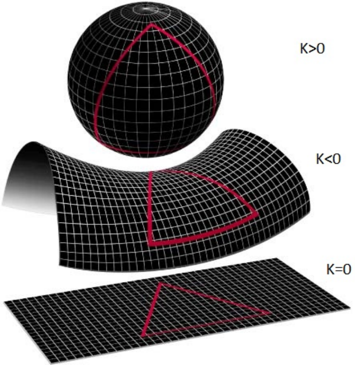
and equality holds if and only if Ω is isometric to an n-dimensional ball.

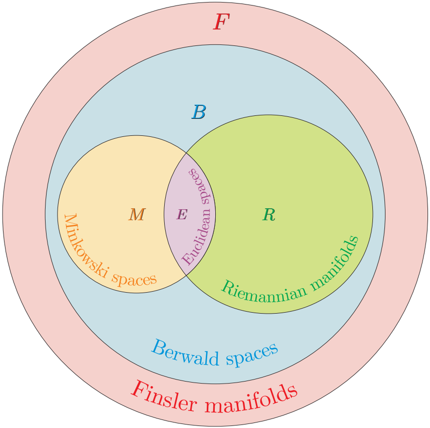
Optimal isoperimetric profiles in $\mathbb{R}^n$: balls!



Nature prefers symmetric/perfect objects.

What happens in curved spaces?





Global approach: Riemannian/Finsler manifolds

- orbits of aircrafts (geodesics on curved spaces);
- general relativity (Einstein's equation, gravitation vs. curvature).

(M, g): an n-dimensional Riemannian manifold.

- **Negatively curved spaces** (Cartan-Hadamard conjecture): $\mathsf{K} \leq 0$
  for every bounded open set $\Omega \subset M$ with smooth boundary, one has

$$\mathcal{P}(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \text{Vol}(\Omega)^{\frac{n-1}{n}}.$$

  The conjecture holds for $n = 2$, 3 and 4. Open for $n \geq 5$.

- **Positively curved spaces**[1]: $\mathsf{Ric} \geq 0$
  for every bounded open set $\Omega \subset M$ with smooth boundary, one has

$$\mathcal{P}(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathsf{AVR}^{\frac{1}{n}} \text{Vol}(\Omega)^{\frac{n-1}{n}}; \quad (1)$$

  - $\mathsf{AVR} = \lim_{r\to\infty} \frac{\text{Vol}(B(x_0, r))}{\omega_n r^n} \in [0, 1]$: asymptotic volume ratio.
  - (1) is sharp and we have a strong curvature rigidity:
    equality holds in (1) for some $\Omega \subset M$ if and only if (M, g) is isometric to $\mathbb{R}^n$ ($\mathsf{AVR} = 1$) and Ω is isometric to a ball in $\mathbb{R}^n$.

[1] S. Brendle (CPAM, 2021), Z. Balogh & A. Kristály (Math. Annalen, 2021)

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Conjecture. (Lord Rayleigh, 1877, *The Theory of Sounds*)
The disc has the minimal fundamental tone among *clamped plates* with a given area.

Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be bounded and $\Omega^\star \subset \mathbb{R}^n$ be a ball with $\mathrm{Vol}(\Omega^\star) = \mathrm{Vol}(\Omega)$.

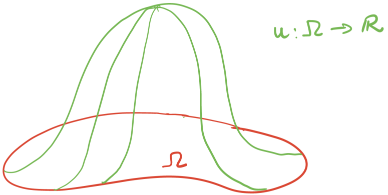
- **Clamped plate problem**:

$$\begin{cases} \Delta^2 u = \Lambda_1(\Omega) u & \Omega, \\ u = |\nabla u| = 0 & \partial\Omega. \end{cases}$$

  - Δ: Laplace operator.

  Conjecture:

$$\Lambda_1(\Omega) \geq \Lambda_1(\Omega^\star) = \mathfrak{h}_\nu^4 \left(\frac{\omega_n}{\mathrm{Vol}(\Omega)} \right)^{\frac{4}{n}}.$$



- $\mathfrak{h}_\nu$: the first critical point of J_ν / I_ν.
- $\nu := \frac{n}{2} - 1$.

- **Variational characterization**: $\Lambda_1(\Omega) = \inf_{u \in W_0^{2,2}(\Omega) \setminus \{0\}} \dfrac{\int_\Omega (\Delta u)^2 \mathrm{d}x}{\int_\Omega u^2 \mathrm{d}x}$.
- Szegő (1950): $\pi_1(\Omega) = 0$ and u_1 is *sign-preserving* $\implies \Lambda_1(\Omega) \geq \Lambda_1(\Omega^\star)$.
- Coffman–Duffin–Schaffer ($\sim$ 1952): Let $0 < r < R$ and the annulus

$$A_{r,R} = \{x \in \mathbb{R}^2 : r < |x| < R\}.$$

  - $\mathsf{C}_{CDS} \approx 762.3264$;
  - u_1 is **sign-preserving** if $R/r < \mathsf{C}_{CDS}$, and **sign-changing** if $R/r > \mathsf{C}_{CDS}$.

- **Landmark results**: Lord Rayleigh's conjecture confirmed in *2 and 3 dimensions* by Ashbaugh-Benguria (1995) and Nadirashvili (1995).
- No results for higher dimensions ($n \geq 4$) and in curved spaces.

What happens with the clamped plates in curved spaces?

Let (M,g) be an n-dimensional complete Riemannian manifold, $n \geq 2$.

- Clamped plate problem: $\Omega \subset M$ is any bounded open domain,

$$\begin{cases} \Delta_g^2 u = \Lambda u & \text{in} \quad \Omega, \\ u = |\nabla_g u| = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- Variational characterization (calculus of variations):

$$\Lambda_g(\Omega) = \inf_{u \in W_0^{2,2}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega (\Delta_g u)^2 \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \mathrm{d}v_g}.$$

- **Strong curvature dependence**:
  - **Nonpositively curved spaces:**
    A. Kristály, Fundamental tones of clamped plates in nonpositively curved spaces, *Adv. Math.*, 2020.
  - **Positively curved spaces:**
    A. Kristály, Lord Rayleigh's conjecture for positively curved vibrating clamped plates, *Geom. Funct. Anal.*, 2022.

Nonpositively curved spaces[2]**:**

Let $n \in \{2, 3\}$ and (M, g) be an n-dimensional Cartan-Hadamard manifold with sectional curvature $\mathsf{K} \leq -\kappa$ for some $\kappa \geq 0$, $\Omega \subset M$ be a bounded domain with smooth boundary and volume $V_g(\Omega) \leq \frac{c_n}{\kappa^{n/2}}$ with $c_2 \approx 21.031$ and $c_3 \approx 1.721$. If $\Omega^\star \subset N_\kappa^n$ is a geodesic ball verifying $V_g(\Omega) = V_\kappa(\Omega^\star)$ then

$$\Gamma_g(\Omega) \geq \Gamma_\kappa(\Omega^\star), \tag{2}$$

with equality in (2) if and only if Ω is isometric to $\Omega^\star$. In addition,

$$\Gamma_g(\Omega) \geq \frac{(n-1)^4}{16}\kappa^2. \tag{3}$$

Curvature implication: in negatively curved spaces, arbitrary big clamped drums have positive fundamental tones, see (3), contrary to the Euclidean framework!

[2] A. Kristály (Adv. Math., 2020)

Positively curved spaces[3]**:**

Let (M, g) be a compact n-dimensional Riemannian manifold with $\mathrm{Ric} \geq (n-1)\kappa > 0$ where $n \geq 2$. Then there exists $v_n \in [0, 1)$ (not depending on $\kappa > 0$), with $v_2 = v_3 = 0$ and $v_n > 0$ for $n \geq 4$ such that if $\Omega \subset M$ is a smooth domain and $\Omega^\star \subset \mathbb{S}^n_\kappa$ is a spherical cap with $\frac{V_g(\Omega)}{V_g(M)} = \frac{V_\kappa(\Omega^\star)}{V_\kappa(\mathbb{S}^n_\kappa)} > v_n$, then

$$\Lambda_g(\Omega) \geq \Lambda_\kappa(\Omega^\star). \tag{4}$$

Equality holds in (4) if and only if (M, g) is isometric to $(\mathbb{S}^n_\kappa, g_\kappa)$ and Ω is isometric to $\Omega^\star$. In addition, $v_\infty = \limsup_{n \to \infty} v_n < 1$.

Curvature implication: the first result which provides an affirmative answer to Lord Rayleigh's conjecture in high dimensions!

[3] A. Kristály (Geom. Funct. Anal., 2022)

I. PDEs and differential inclusions via critical point theory: various phenomena from mathematical physics (clamped plates, fixed membranes) and engineering (elliptic differential inclusions), etc.

II. Anisotropic isoperimetric problems: the famous conjecture of Cabré, Ros-Oton & Serra on Finsler (non-Riemannian) manifolds, where the expected optimal isoperimetric profiles are the Wulff-shapes (and NOT the usual balls)!

- Model objects (no symmetry): black holes (via Finsler-Poincaré disc), walking law under the action of gravity on a mountain slope.
- Wulff-shapes instead of usual balls: convex limaçons (balls = shape of eggs)!

III. Sharp functional inequalities: entropy and hypecontractivity estimates via sharp logarithmic Sobolev inequalities on metric measure spaces curved in the sense of Lott-Sturm-Villani.

- **Program coordinator** (Senior category):
  Functional inequalities and elliptic PDEs: the influence of curvature. 2018–2022, NKFIH, K_18, No. 127926.
  - Cs. Farkas (my former PhD Student, ÓE);
  - Á. Mester (current PhD Student, ÓE).

- **PhD supervising** (AIAMDI, Óbuda University):
  - Á. Mester: isoperimetric inequalities, functional inequalities on Finsler manifolds, defense in 2022-2023;
  - K. Szilák: differential inclusions, engineering problems;
  - B. Oltean-Peter: equilibrium problems on curved spaces.

- Several **D1 publications** with my PhD Students.

- Work with the research group from the **Mathematical Institute of Bern, Switzerland** (Z. Balogh, S. Don, F. Tripaldi).
- *1st semester*. Invitation to the **Fields Institute, Toronto, Canada**, November 2022; thematic week (*Workshop on Aspects of Ricci Curvature Bounds*) and another week for collaborations and research.
- *2nd semester*. Invitation to **Oxford University, UK**, collaboration with professor A. Mondino, April 2023.
- *2nd semester*. Invitation to **Hangzhou University, China**, June-July 2023, giving special lectures within a one-week Summer School.

- **Scientific impact:**
  - mainstream research direction: optimal mass transport (Fields medal in 2018 for A. Figalli), isoperimetric conjectures;
  - invitations to visit top universities (Bern, Oxford, Fields Institute);
  - citations from top mathematicians (Princeton, Oxford, ETH Zürich, etc), including Fields medalists.
- **Social/Economical impact**:
  - application of optimal mass transport theory;
  - monopolist-agent problem.
- **Forming a competitive team/generation at ÓE** in **applied mathematics** (NKFIH grant + Excellence Program).

About me...

- Challenging optimization problem: Family & Mathematics.
- 4 children: Boy + Girl + Girl + Boy.
- Hungarian folk dance (due to my wife) + playing several instruments.
- On the beach (*splashing the water* versus *isoperimetric problems*):