Three fuzzy problem(s)

Ivana Štajner-Papuga
Department of Mathematics and Informatics, Faculty of Sciences
University of Novi Sad, Serbia
ivana.stajner-papuga@dim.uns.ac.rs

John von Neumann

If people do not believe that **mathematics is simple**, it is only because they do not realize how **complicated life** is.

Three fuzzy problem(s)

- **fuzzy** in fuzzy sets

Three **fuzzy** problem(s)

- ✿ **fuzzy** in fuzzy sets
- ✿ **fuzzy** in fuzzy measure

Three **fuzzy** problem(s)

- ✿ **fuzzy** in fuzzy sets
- ✿ **fuzzy** in fuzzy measure
- ✿ **fuzzy** in fuzzy integral

Fuzzy sets

Classical set - characteristic function

$$\chi_A : X \to \{0, 1\} \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Fuzzy sets

Classical set - characteristic function

$$\chi_A : X \to \{0, 1\} \qquad \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Fuzzy set - membership function

$$\mu_A : X \to [0, 1]$$

Fuzzy sets

Fuzzy set - membership function

$$\mu_A : X \to [0, 1]$$

L. A. Zadeh, "Fuzzy sets," Information and Control, vol. 8, no. 3, pp. 338-353, June 1965.

https://www2.eecs.berkeley.edu/Faculty/Homepages/zadeh.html

Fuzzy measure

Let X be the universal set, let $\mathcal{D}$ be a family of subsets of X that contains empty set.
The **classical measure** is a set function m on $\mathcal{D}$ that is

- nonnegative,
- maps empty set to zero,
- σ-additive.

Fuzzy measure

Let X be the universal set, let $\mathcal{D}$ be a family of subsets of X that contains empty set.
The **classical measure** is a set function m on $\mathcal{D}$ that is

- nonnegative,
- maps empty set to zero,
- σ-additive.

1898. - **E. Borel**

1900. - **H. Lebesgue**

1933. - **A. N. Kolmogorov**

Fuzzy measures - why?

1961.- The Ellsberg paradox

Gamble A. Draw a marble from an urn known to contain x red and y black marbles such that x+y=100.
Gamble B. Draw a marble from an urn known to contain 50 red and 50 black marbles.

Fuzzy measures - why?

1961.- The Ellsberg paradox

Gamble A. Draw a marble from an urn known to contain x red and y black marbles such that x+y=100.
Gamble B. Draw a marble from an urn known to contain 50 red and 50 black marbles.

$$(A,r) \sim (A,b) \qquad (B,r) \sim (B,b)$$
$$(B,r) \succ (A,r) \qquad (B,b) \succ (A,b)$$

Fuzzy measures - why?

1961.- The Ellsberg paradox

Gamble A. Draw a marble from an urn known to contain x red and y black marbles such that x+y=100.
Gamble B. Draw a marble from an urn known to contain 50 red and 50 black marbles.

$$(A, r) \sim (A, b) \qquad (B, r) \sim (B, b)$$
$$(B, r) \succ (A, r) \qquad (B, b) \succ (A, b)$$

$$P(A, r) = P(A, b) = \frac{1}{2} \qquad P(B, r) = P(B, b) = \frac{1}{2}$$

Fuzzy measures - why?

1961.- The Ellsberg paradox

Gamble A. Draw a marble from an urn known to contain x red and y black marbles such that x+y=100.
Gamble B. Draw a marble from an urn known to contain 50 red and 50 black marbles.

$$(A, r) \sim (A, b) \qquad (B, r) \sim (B, b)$$
$$(B, r) \succ (A, r) \qquad (B, b) \succ (A, b)$$

$$P(A, r) = P(A, b) = \frac{1}{2} \qquad P(B, r) = P(B, b) = \frac{1}{2}$$

$$P(B, r) > P(A, r) \qquad P(B, b) > P(A, b)$$

Fuzzy measures - why?

1961.- The Ellsberg paradox

Gamble A. Draw a ball from an urn known to contain x red and y black balls such that x+y=100.
Gamble B. Draw a ball from an urn known to contain 50 red and 50 black balls.

$$(A, r) \sim (A, b) \qquad (B, r) \sim (B, b)$$
$$(B, r) \succ (A, r) \qquad (B, b) \succ (A, b)$$

$$\mathbf{P}(\mathbf{A}, \mathbf{r}) = \mathbf{P}(\mathbf{A}, \mathbf{b}) = \frac{\mathbf{1}}{\mathbf{2}} \qquad P(B, r) = P(B, b) = \frac{1}{2}$$

$$P(B, r) > P(A, r) \qquad P(B, b) > P(A, b)$$

Fuzzy measures - how?

1953. - G. Choquet, "Theory of Capacities", Annales de l'Institut Fourier 5: 131-295.

1974. - M. Sugeno, "Theory of fuzzy integrals and its applications", Ph.D. thesis, Tokyo Institute of Technology, Tokyo, Japan.

additivity → monotonicity

Fuzzy measures - how?

1953. - G. Choquet, "Theory of Capacities", Annales de l'Institut Fourier 5: 131-295.

- increasing

$$A \subseteq B \quad \text{then} \quad m(A) \leq m(B)$$

- strongly subadditive

$$m(A \cup B) < m(A) + m(B), \qquad A \cap B = \emptyset$$

- continuous

Fuzzy measures - how?

1974. - M. Sugeno, "Theory of fuzzy integrals and its applications", Ph.D. thesis, Tokyo Institute of Technology, Tokyo, Japan.

- vanishing at the emptyset
- increasing
- continuous from below
- continuous from above

Fuzzy measures - how?

1967. - A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping", The Annals of Mathematical Statistics

1976. - G. Shafer, "A Mathematical Theory of Evidence", Princeton University Press

belief measures $Bel : \mathcal{P}(X) \to [0,1]$:

additivity $\to$ superadditivity

$$Bel(A \cup B) \geq Bel(A) + Bel(B), \qquad A \cap B = \emptyset$$

plausibility measures $Pl : \mathcal{P}(X) \to [0,1]$:

additivity $\to$ subadditive

$$Pl(A \cup B) \leq Pl(A) + Pl(B), \qquad A \cap B = \emptyset$$

Fuzzy measures - how?

1978. - L. Zadeh, "Fuzzy Sets as the Basis for a Theory of Possibility", Fuzzy Sets and Systems 1, 3-28.

1988 - D. Dubois and H. Prade, "Possibility Theory", Plenum Press, New York.

possibility measures $\pi : \mathcal{P}(X) \to [0,1]$:

$$\pi(X) = 1 \qquad \text{and} \qquad \pi(\cup_{i\in I} A_i) = \textbf{sup}_{i\in I}\pi(A_i)$$

necessity measures $\nu : \mathcal{P}(X) \to [0,1]$:

$$\nu(\emptyset) = 0 \qquad \text{and} \qquad \nu(\cap_{i\in I} A_i) = \inf_{i\in I} \nu(A_i)$$

S-measure

A **triangular conorm** S is a function $S : [0,1]^2 \to [0,1]$ such that

- $S(x, y) = S(y, x)$,
- $S(x, S(y, z)) = S(S(x, y), z)$,
- $S(x, y) \leq S(x, z)$ fr $y \leq z$,
- $S(x, 0) = x$.

Let X be the universal set, let Σ be a σ-algebra of subsets of X. A mapping $\mu : \Sigma \to [0,1]$ is a **S-measure** if $\mu(\emptyset) = 0$, and for all $A, B \in \Sigma$ such that $A \cap B = \emptyset$ holds

$$\mu(A \cup B) = S(\mu(A), \mu(B)).$$

E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms.* Dordrecht: Kluwer Academic Publishers 2000.

$\oplus$-measure

Let $[a, b]$ be a closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be a total order on $[a, b]$. A **semiring** is a structure $([a, b], \oplus, \odot)$ such that the following holds:

- $\oplus$ is **pseudo-addition**, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;
- $\odot$ is **pseudo-multiplication**, i.e., a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing $\left(x \preceq y \text{ implies } x \odot z \preceq y \odot z, \right.$ $\left. z \in [a, b]_+ = \{x : x \in [a, b], \ \mathbf{0} \preceq x\} \right)$, associative and for which exists a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

$\oplus$-measure

A set function $\mu : \Sigma \to [a,b]_+$ is a **σ-$\oplus$-measure** if

i) $\mu(\emptyset) = \mathbf{0}$,

ii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \mu(A_i)$,

where $(A_i)_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint sets from Σ.

$$\mu(A \cup B) = \mu(A) \oplus \mu(B)$$

E. Pap, *Pseudo-additive measures and their applications.* In: Handbook of Measure Theory (E. Pap, ed.), Volume II, pp. 1403-1465, Elsevier, North-Holland 2002.

Fuzzy measures - in general

$$\textbf{fuzzy measures} \quad \leftrightarrow \quad \textbf{monotone set functions}$$

Fuzzy measures - in general

$$\text{fuzzy measures} \quad \leftrightarrow \quad \text{monotone set functions}$$

Let X be the universal set, let $\mathcal{D}$ be a family of subsets of X that contains empty set.
A mapping $\mu : \mathcal{D} \to [0, \infty)$ is a **fuzzy measure** if

- $\mu(\emptyset) = 0$,
- if $A \subseteq B$ then
$$\mu(A) \leq \mu(B).$$

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

monotone set function $\rightarrow$ **the Sugeno integral**

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

monotone set function $\rightarrow$ **the Sugeno integral**

S-measure $\rightarrow$ **the** (S, T)**-integral**

S-measure $\rightarrow$ **the** (S, U)**-integral**

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

monotone set function $\rightarrow$ **the Sugeno integral**

S-measure $\rightarrow$ **the (S,T)-integral**

S-measure $\rightarrow$ **the (S,U)-integral**

$\oplus$-measure $\rightarrow$ **the pseudo-integral**

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

monotone set function $\rightarrow$ **the Sugeno integral**

S-measure $\rightarrow$ **the (S,T)-integral**

S-measure $\rightarrow$ **the (S,U)-integral**

$\bigoplus$-measure $\rightarrow$ **the pseudo-integral**

the general fuzzy integral

Fuzzy integration

monotone set function $\rightarrow$ **the Choquet integral**

monotone set function $\rightarrow$ **the Sugeno integral**

S-measure $\rightarrow$ **the (S, T)-integral**

S-measure $\rightarrow$ **the (S, U)-integral**

$\oplus$-measure $\rightarrow$ **the pseudo-integral**

the general fuzzy integral

the universal integral

Fuzzy integration

monotone set function → **the Choquet integral**

monotone set function → **the Sugeno integral**

S-measure → **the (S, T)-integral**

S-measure → **the (S, U)-integral**

$\oplus$-measure → **the pseudo-integral**

the general fuzzy integral

the universal integral - **one to rule them all**

Example I

1995. - M. Grabisch, H. T. Nguyen, E. A. Walker, *Fundamentals of Uncertainty Calculi with Applications to Fuzzy Interence,* Kluwer Academics Publishers, Dordrecht.

Example I

1995. - M. Grabisch, H. T. Nguyen, E. A. Walker, *Fundamentals of Uncertainty Calculi with Applications to Fuzzy Interence*, Kluwer Academics Publishers, Dordrecht.

Student A: $M = 18$, $F = 16$, $L = 10$;
Student B: $M = 10$, $F = 12$, $L = 18$;
Student C: $M = 14$, $F = 15$, $L = 15$.

1. $\mu(\{M\}) = \mu(\{F\}) = 0.45$, $\mu(\{L\}) = 0.3$;
2. $\mu(\{M, F\}) = 0.5 < \mu(\{M\}) + \mu(\{F\}) = 0.9$;
3. $\mu(\{M, L\}) = \mu(\{F, L\}) = 0.9 > \mu(\{M\}) + \mu(\{L\}) = 0.75$.
4. $\mu(\{M, F, L\}) = 1$, $\mu(\emptyset) = 0$.

Example I

The Choquet integral of an arbitrary simple function $f : X \to \{\omega_1, \omega_2, \ldots, \omega_n\}$, based on a fuzzy measure μ, is:

$$(C) \int_X f \, d\mu = \sum_{i=1}^{n} (\omega_i - \omega_{i-1}) \cdot \mu(\Omega_i)$$

where $\Omega_i = \{x \mid f(x) \geq \omega_i\}$ and $\omega_0 = 0$.

Example I

The Choquet integral of an arbitrary simple function $f : X \to \{\omega_1, \omega_2, \ldots, \omega_n\}$, based on a fuzzy measure μ, is:

$$(C)\int_X f\, d\mu = \sum_{i=1}^{n} (\omega_i - \omega_{i-1}) \cdot \mu(\Omega_i)$$

where $\Omega_i = \{x \mid f(x) \geq \omega_i\}$ and $\omega_0 = 0$.

	A	B	C
arithmetic men	15.25	12.75	14.625
Choquet integral	13.9	13.6	14.9

Examples - Maximal Covering Location Problem

- $X = \{L_1, L_2, \ldots, L_R\}$ - set of all locations;
- $Y = \{Y_1, Y_2, \ldots, Y_P\}$ - set of all facilities;
- $\mu : \mathcal{P}(Y) \to [0, 1]$ - measure of interaction for different facilities modelled by a **fuzzy measure**;
- $\omega_{i,j} \in [0, 1]$ - degree of coverage for location L_i by the j-th facility.

$$f_{L_i} : Y \to \{\omega_{i,1}, \omega_{i,2}, \ldots, \omega_{i,m}\}, \quad i = 1, \ldots, R;$$

Coverage degree

$$g = \sum_i \mathbf{(C)} \int \mathbf{f_{L_i} d}\mu;$$

A. Takači et al.: An Extension of Maximal Covering Location Problem based on the Choquet Integral, Acta Polytechnica Hungarica, Vol 13, 205-220, 2016.

Example - Impact Evaluation of European Capital of Culture - ECoC index

- The outer level of aggregation observes three basic segments dimensions that participate in the formation of the ECoC index, namely Culture, Economy, and Community. The relevance of the observed segment is expressed by a **fuzzy measure called the measure of significance** that is predefined by experts. Index is calculated by **Choquet integral**.
- The inner level of aggregation focuses on calculation of an index for each segment segment (Culture, Economy, Community) and it is also done by **Choquet integral** with respect to **measure of significance** on subsegments.

ECoC(2018)=6.33 **ECoC(2019)=6.7575** **ECoC(2020)=6.283**
ECoC(2021)=6.4655 **ECoC(2021)=8.412**

M. Vuičić et al.:Prepare for Impact! A Methodological Approach for Comprehensive Impact Evaluation of European Capital of Culture: The Case of Novi Sad 2022, Social Indicators Research, Vol 165, 715-736, 2023.

Example - Choquet Integral in Ranking Crimes

Step 1: Modeling attributes of the data are done by **fuzzy sets**.

Step 2: Simple functions for all data are formed by extracting numerical values from the fuzzy set in Step 1.

Step 3: The importance and interaction of attributes of the data are described by **fuzzy measures** obtained using aggregation operators on singleton values determined by experts.

Step 4: Each data is evaluated by an integral aggregation operator (**fuzzy integral**) based on the fuzzy measure in Step 3.

M. E. Cornejo et al.: On Choquet integral in ranking crimes. Studies in Computational Intelligence. Springer. Studies in Computational 1040, 181-187, 2023.

Example - Statistical notions

- "Fuzzy to crisp"
- "Crisp to fuzzy"
- **"Fuzzy to fuzzy"**

Example - Statistical notions

- "Fuzzy to crisp"

Fuzzy measure as a base for monotone expectation

The **monotone expectation** of f with respect to μ, denoted with E_μ, is

$$E_\mu(f) = (C)\int_X f\, d\mu.$$

De Campos, L.M., & Bolaños, M.J. (1989). Representation of fuzzy measures through probabilities. Fuzzy Sets and Systems, 31, 23-36.

Reche, F., Morales, M., & Salmerón, A. (2020). Statistical Parameters Based on Fuzzy Measures. Mathematics, 8(11), 2015.

Example - Statistical notions

- "Fuzzy to crisp"

Fuzzy sets in statistical hypotheses

"softening" statistical hypotheses - assumptions are no longer crisp
The **fuzzy hypothesis** is

$$\mathbf{FH}: \theta \text{ is } A, \text{ where } A \text{ is a fuzzy subset of } \Theta.$$

Parchami, A., Taheri, S.M., & Mashinchi, M. (2010). Fuzzy p-value in testing fuzzy hypotheses with crisp data. Statistical Papers, 51, 209.

Example - Statistical notions

- "Crisp to fuzzy"

Possibilistic expectation of fuzzy numbers

The **possibilistic expected value of a fuzzy number** A is

$$E(A) = \int_0^1 2\alpha E(U_{A,\alpha})\, d\alpha$$

$E(U_{A,\alpha})$ is the expected value of a random variable $U_{A,\alpha}$ with the uniform probability distribution on **α-cuts** $[a_l(\alpha), a_r(\alpha)]$

Carlsson, C., & Fuller, R. (2001). On possibilistic mean value and variance of fuzzy numbers. Fuzzy Sets and Systems, 122, 315-326.

Example - Statistical notions

- "Crisp to fuzzy"

Fuzzy data in statistical hypotheses

The **statistical test** problem is

$$\mathbf{FH}_0 : \tilde{\theta} = \tilde{\theta}_0 \quad \text{vs} \quad \mathbf{FH}_1 : \tilde{\theta} > \tilde{\theta}_0,$$

where $\tilde{\theta}_0$ is a fixed fuzzy number, and **comparison of fuzzy numbers is done through α-cuts**

$$\tilde{\theta} > \tilde{\theta}_0 \quad \text{iff} \quad \tilde{\theta}_l(\alpha) > (\tilde{\theta}_0)_l(\alpha) \quad \text{and} \quad \tilde{\theta}_r(\alpha) > (\tilde{\theta}_0)_r(\alpha)$$

Wu, H.C. (2005). Statistical hypotheses testing for fuzzy data. Information Sciences, 279, 446-459.

Example - Statistical notions

- **"Fuzzy to fuzzy"**

Expected value of fuzzy events through fuzzy integrals

- **Choquet expectation**

$$(C)\mathbf{E}_\mu(A) = (C)\int m_A \, d\mu = \int_0^1 \mu([A]^\alpha) d\alpha$$

- **Sugeno expectation**

$$(S)\mathbf{E}_\mu(A) = (S)\int m_A \, d\mu = \sup_{\alpha \in [0,1]} \min(\alpha, \mu([A]^\alpha))$$

Klement, E. P., & Mesiar, R. (2015). On the Expected Value of Fuzzy Events. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 23, 57-74.

Example - Statistical notions

- **"Fuzzy to fuzzy"**

Possibilistic expected value of fuzzy events through fuzzy integrals

The **GFI based expected value** *for* A*, and is given by*

$$M_{GFI}(A) = \int_X^{\oplus} \alpha E(U_{A,\alpha}) \otimes d\mu,$$

where $E(U_{A,\alpha})$ *is the expectation of a random variable* $U_{A,\alpha}$ *with uniform distribution on* $[a_l(\alpha), a_r(\alpha)]$.

Grujić, G., Lozanov-Crvenković, Z., & Štajner-Papuga, I. (2017). General fuzzy integral as a base for estimation of fuzzy quantities. Fuzzy Sets and Systems, 326, 69-80.

Example - Statistical notions

- **"Fuzzy to fuzzy"**

Horizontal fuzzy relations and hypotheses testing

Let the universe X be the real line, and let $\mathcal{F}$ be the family of all triangular fuzzy numbers.

$$S_L(A,B) = \begin{cases} 0, & LTR(A) \subseteq LTR(B) \\ \sup_{\{x \mid m_{LTR(A)(x)} > m_{LTR(B)}(x)\}} m_B(x), & \textit{otherwise} \end{cases}$$

$$FL_{AB} = \begin{cases} S_L(A,B) & S_L(A,B) \le S_L(B,A) \\ 1 - S_L(B,A) & \textit{otherwise} \end{cases}$$

A. Takači et al.: On Horizontal Fuzzy Relations and Hypotheses Testing, Acta Polytechnica Hungarica, Vol 21, 153-166, 2024.

Example - Statistical notions

- **"Fuzzy to fuzzy"**

Horizontal fuzzy relations and hypotheses testing

Let the universe X be the real line, and let $\mathcal{F}$ be the family of all triangular fuzzy numbers.

The **horizontal fuzzy relation** $\preceq_F$, *where* $\preceq_F \colon \mathcal{F} \times \mathcal{F} \to [0,1]$, *is*

$$\preceq_F (A, B) = A \preceq_F B = 0.5(FL_{AB} + FR_{AB}).$$

A. Takači et al.: On Horizontal Fuzzy Relations and Hypotheses Testing, Acta Polytechnica Hungarica, Vol 21, 153-166, 2024.

Example - Statistical notions

- **"Fuzzy to fuzzy"**

Horizontal fuzzy relations and hypotheses testing

Let the universe X be the real line, and let $\mathcal{F}$ be the family of all triangular fuzzy numbers.

The acceptance degree of a hypothesis $\mathbf{FH}_0$ *is*

$$AD_{\mathbf{FH}_0} = \max(\tilde{\theta}_0 - \hat{y} \preceq_F \frac{z_{\xi/2}}{\sqrt{n}} \cdot \tilde{\sigma}, \frac{-z_{\xi/2}}{\sqrt{n}} \cdot \tilde{\sigma} \preceq_F \tilde{\theta}_0 - \hat{y}),$$

where $\hat{y}$ is fuzzy arithmetic mean of $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$, $z_{\xi/2}$ is $1 - \frac{\xi}{2}$-quantile of normal $\mathcal{N}(0,1)$ distribution.

A. Takači et al.: On Horizontal Fuzzy Relations and Hypotheses Testing, Acta Polytechnica Hungarica, Vol 21, 153-166, 2024.

… and many more

... and many more

There is something fuzzy in complicated life. ☺