Kinematics of Delta-type Parallel Robot Mechanisms via Screw Theory: A tutorial paper

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Outline

- Screw theory, Motor algebra
- Geometric method
- 3D movement

- A Reference frame
- Points in the reference frame: $\boldsymbol{P}$ and $\boldsymbol{O}$
- Velocity state of point $\boldsymbol{O}$

The velocity state of one point on the rigid body relative to another is described by:

$$\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{v}_{PO} = \boldsymbol{v}_O + {}^{A}\boldsymbol{\omega}^{B} \times \boldsymbol{r}_{P/O}$$

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where:

- $\boldsymbol{v_P}$ is a point on the body, where the velocity is unknown;
- $\boldsymbol{v_O}$ is the known velocity on the body;
- $\boldsymbol{v_{PO}}$ is the velocity difference between the two points in question;
- ${}^A\boldsymbol{\omega}^B$ is the known angular velocity of the body;
- $\boldsymbol{r_{P/O}}$ is the vector between O and P; if the two points are given in the same frame, then $\boldsymbol{r_{P/O}} = \boldsymbol{P} - \boldsymbol{O}$.

$$\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{v}_{PO} = \boldsymbol{v}_O + {}^A\boldsymbol{\omega}^B \times \boldsymbol{r}_{P/O}$$

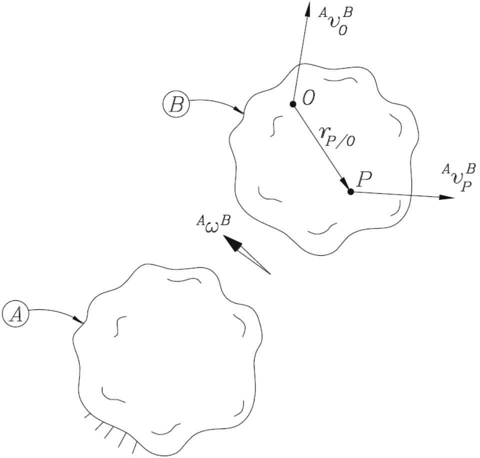


FIGURE: Velocity relations on a rigid body

The basics of Screw theory lie within the understanding of Plücker coordinates. Plücker coordinates represent a line in space as part of $se(3)$ with six coordinates: three representing the line's orientation, and three representing a point on the line[1]. With these values known, a line in three dimensions can be described. There are two representations for the screws:

$$\$ = \begin{bmatrix} \boldsymbol{e}_{[3\times1]} \\ \boldsymbol{m}_{[3\times1]} \end{bmatrix} \qquad (1) \qquad\qquad \$ = \begin{bmatrix} \boldsymbol{m}_{[3\times1]} \\ \boldsymbol{e}_{[3\times1]} \end{bmatrix} \qquad (2)$$

We use the notation of 1.

The screw is represented mathematically as:

$$\$_{[6\times1]} = \left[\boldsymbol{e}_{[3\times1]}, \boldsymbol{m}_{[3\times1]}\right]^T \tag{3}$$

Where:

- $\boldsymbol{e}$ is a vector depicting the direction of the Instantaneous Screw Axis (ISA). It is also known as primary part. In the application of robot kinematics calculation this is aligned with the axis of the actuator.
- $\boldsymbol{m}_{[3\times1]}$ is the dual part of the screw, also known as the moment of a line in the reference frame. (Note: this is called moment, but no relation is shown here between the moment known in mechanics.)

To calculate the dual part of the screw we use

$$\boldsymbol{m} = h\boldsymbol{e} + \boldsymbol{r} \times \boldsymbol{e} \tag{4}$$

$$\boldsymbol{m} = h\boldsymbol{e} + \boldsymbol{r}_{P_2/P_1} \times \boldsymbol{e} \tag{5}$$

where $\boldsymbol{r}$ is a vector pointing from P_1 to P_2 on the body and h is the pitch of the screw (the amount of linear movement performed along the axis, while one complete rotation is performed by the body).

One of the simplest forms of the motion is when

$$\lim_{h\to\infty} h\boldsymbol{e} + \boldsymbol{r} \times \boldsymbol{e}, \quad \omega = 0$$

which results in

$$\$ = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{m} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{e}h \end{bmatrix} \tag{6}$$

This describes a pure linear displacement by h along $\boldsymbol{e}$ with 0 rotation on an infinitely long line.

The other simple form is when

$$h = 0, \quad \omega \neq 0$$

which results in

$$\$ = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{m} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{r} \times \boldsymbol{e} \end{bmatrix} \tag{7}$$

which describes a purely rotational motion, with no linear displacement along the ISA.

A special representation of screws is the unit screw when $|\boldsymbol{e}| = 1$. This is represented by

$$\hat{\$} = \begin{bmatrix} \hat{\boldsymbol{e}} \\ \hat{\boldsymbol{m}} \end{bmatrix} \tag{8}$$

where

- $\hat{\boldsymbol{e}} = \frac{\boldsymbol{e}}{|\boldsymbol{e}|}$
- $\hat{\boldsymbol{m}} = h\hat{\boldsymbol{e}} + \boldsymbol{r} \times \hat{\boldsymbol{e}}$

Klein Form:

$$\{*;*\} : e(3) \times e(3) \to \Re$$

$$\left\{\$^1 \$^2\right\} = \{(\boldsymbol{e}_1 ; \boldsymbol{m}_1), (\boldsymbol{e}_2 ; \boldsymbol{m}_2)\} = \boldsymbol{e}_1 \cdot \boldsymbol{m}_2 + \boldsymbol{e}_2 \cdot \boldsymbol{m}_1 \qquad (9)$$

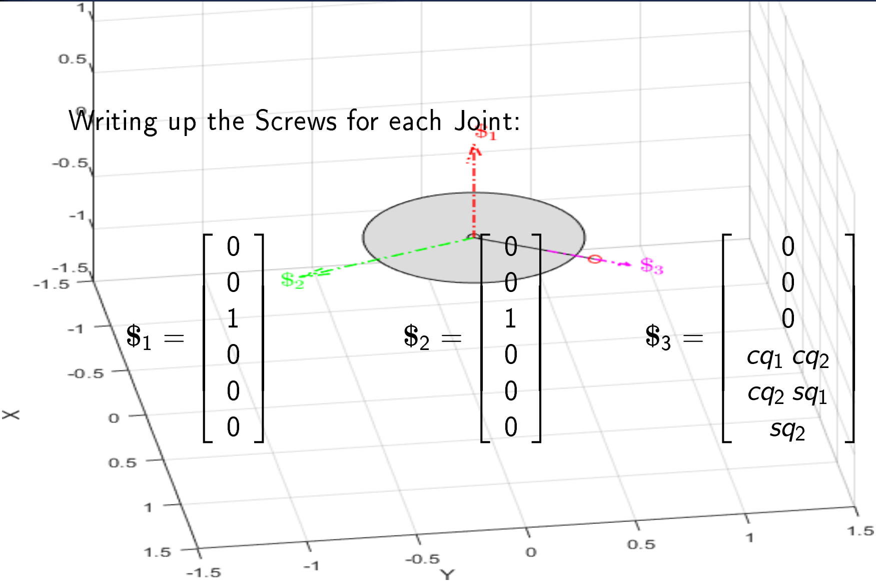
It is said that two screws $\$_1$ and $\$_2$ are reciprocal if the application of the Klein form between them yields $\{\$_1 ; \$_2\} = 0$. The Klein form in other words yields the distance between two screws regardless of their direction. **Note:** In kinematics and dynamics the most often used form is the Klein form, for calculating the passive joint matrices in case of parallel mechanisms.

$$T = \begin{pmatrix} \cos\left(q_1\left(t\right)\right)\cos\left(q_2\left(t\right)\right)\left(l_1 + q_3\left(t\right)\right) \\ \cos\left(q_2\left(t\right)\right)\sin\left(q_1\left(t\right)\right)\left(l_1 + q_3\left(t\right)\right) \\ \sin\left(q_2\left(t\right)\right)\left(l_1 + q_3\left(t\right)\right) \end{pmatrix}$$

Where:

- T - End point of spherical robot
- q_1 - rotational joint position about Z axis
- q_2 - rotational joint position in the plane of XY
- q_3 - linear joint position on the axis of the spherical robot

Writing up the Screws for each Joint:



$$\$_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ cq_1\ cq_2 \\ cq_2\ sq_1 \\ sq_2 \end{bmatrix}$$

Initial equation of moving bodies in Screw Theory, adapted to the spherical manipulator:

$$V^{O}_{[6\times1]} = \begin{bmatrix} \boldsymbol{e}_{[3\times1]} \\ \boldsymbol{m}_{[3\times1]} \end{bmatrix} = \dot{q}_1 \$_1 + \dot{q}_2 \$_2 + \dot{q}_3 \$_3$$

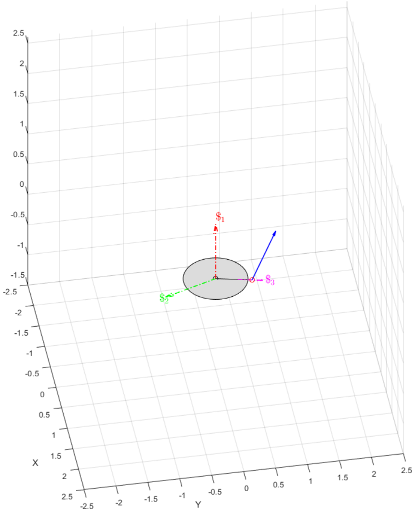
To get the velocity of the endpoint:

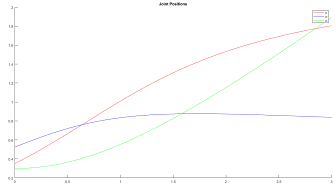
$$\boldsymbol{V} = \boldsymbol{V}^{\boldsymbol{O}}_{\boldsymbol{m}} + \boldsymbol{V}^{\boldsymbol{O}}_{\boldsymbol{e}} \times \boldsymbol{T}$$

Which results in the equation below, where $\boldsymbol{J}$ is the well-known Jacobian

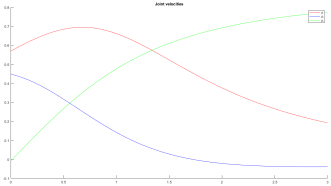
$$\boldsymbol{V} = \underbrace{\begin{bmatrix} -cq_2\, sq_1\, (l_1 + q_3) & -cq_1\, sq_2\, (l_1 + q_3) & cq_1\, cq_2 \\ cq_1\, cq_2\, (l_1 + q_3) & -sq_1\, sq_2\, (l_1 + q_3) & cq_2\, sq_1 \\ 0 & cq_2\, (l_1 + q_3) & sq_2 \end{bmatrix}}_{\boldsymbol{J}} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}}_{\dot{q}}$$

Serial Example
Spherical Manipulator





(a) Joint positions



(b) Joint Velocities

Figure: Spherical Joint variables

Given:

- Basic configuration of the GTPR with all characteristic points ($\boldsymbol{O}, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{T}, q_1, q_2, q_3$)
- Velocity of TCP: $\boldsymbol{v}$

Equation of TCP Velocity:

$$\boldsymbol{v} = {}^{0}_{X}\omega^{1}\,{}^{0}_{X}\hat{\$}^{1}_{O} + {}^{1}_{X}\omega^{2}\,{}^{1}_{X}\hat{\$}^{2}_{O} + {}^{2}_{X}\omega^{3}\,{}^{2}_{X}\hat{\$}^{3}_{O} + \\ + {}^{3}_{X}\omega^{4}\,{}^{3}_{X}\hat{\$}^{4}_{O} + {}^{4}_{X}\omega^{5}\,{}^{4}_{X}\hat{\$}^{5}_{O} + {}^{5}_{X}\omega^{6}\,{}^{5}_{X}\hat{\$}^{6}_{O}$$

where

- X substitution of each respective arm A, B, C
- ${}^{0}_{X}\omega^{1}\,{}^{0}_{X}\hat{\$}^{1}_{O}$ is a virtual screw introduced in order to find the kinematic relationship between the joint-space and tool-workspace easier.
- ${}^{1}_{X}\omega^{2}$ is the angular velocity of the actuator (other notation: $\dot{q}$).

In parallel mechanisms it is necessary to locate a screw which is able to cancel out most of the terms from the equation of the TCP velocity. In this case it is a screw augmented to the passive links of the delta robot. We will denote these screws with: $\$_O^{l_x}$. Applying the Klein-form to the left and right sides of the equation of TCP velocity:

$$\boldsymbol{A}_{[3\times3]}\ \boldsymbol{v}_{[3\times1]} = \boldsymbol{B}_{[3\times3]}\ \dot{\boldsymbol{q}}_{[3\times1]}$$

Where:

- $\boldsymbol{A}$ is called the reduced active matrix of the manipulator.
- $\boldsymbol{B}$ which is called the first-order driver matrix

$$\boldsymbol{A} = \left[\boldsymbol{e}^{l_A}\ \boldsymbol{e}^{l_B}\ \boldsymbol{e}^{l_C}\right]^T$$

in which $\boldsymbol{e}^{l_X} = \boldsymbol{X}_3 - \boldsymbol{X}_2$ is the direction of the passive joint

$$\boldsymbol{B} = \begin{bmatrix} \left\{ {}_A^1\$_O^2 ; \$_O^{l_A} \right\} & 0 & 0 \\ 0 & \left\{ {}_B^1\$_O^2 ; \$_O^{l_B} \right\} & 0 \\ 0 & 0 & \left\{ {}_C^1\$_O^2 ; \$_O^{l_C} \right\} \end{bmatrix}$$

Rearranging:

$$\boldsymbol{A}\,v = \boldsymbol{B}\,\dot{\boldsymbol{q}}$$

to:

$$v = \boldsymbol{A}^{-1}\boldsymbol{B}\dot{\boldsymbol{q}}$$

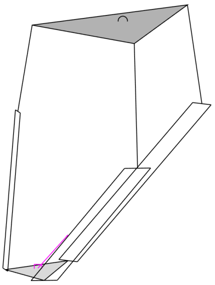
where:

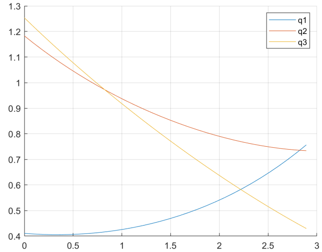
$$\boldsymbol{J} = \boldsymbol{A}^{-1}\boldsymbol{B}$$

so:

$$v = \boldsymbol{J}\dot{\boldsymbol{q}}$$

Which gives the description of differential movement in the current configuration of the robot





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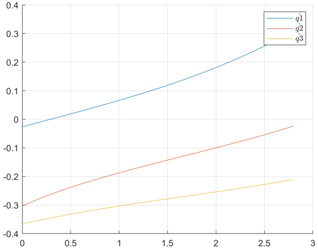

(b) Joint Velocities

R. Featherstone, "Plucker basis vectors," in *Proceedings 2006 IEEE International Conference on Robotics and Automation, 2006. ICRA 2006.*, May 2006, pp. 1892–1897, iSSN: 1050-4729.

Thank you for your attention!

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